Qualitative spatial reasoning I: Crisp contact structures

Ivo Düntsch

Department of Computer Science,
Brock University,
St. Catharines, Ontario
duentsch@brocku.ca

Plan

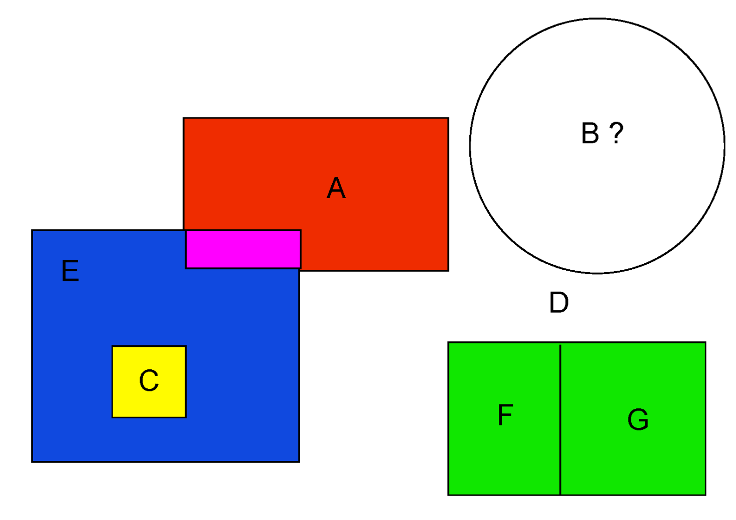
- An example of spatial reasoning
- What is space made of?
- Some musings on pointless geometry/topology.
- A brief overview of the history of “qualitative spatial reasoning” (QSR).
- Relational concepts
- Contact structures
- A review of topological concepts.
- Concrete regions: Regular closed sets.
- Abstract regions: Boolean contact algebras.
- Bringing it all together - a representation theorem.

An example of spatial reasoning: The confederation

1. The confederation consists of exactly 7 countries/provinces (a, b, c, d, e, f, g) on an island.
2. a shares a border with another country.
3. Country a and country d have no common border.
4. Country c is surrounded by country e.
5. Country d consists of two provinces f and g.
6. Country e and a have no common border.

Question: Do a and b share a common border?

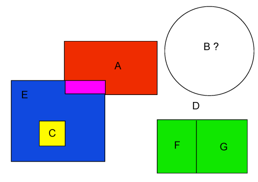
The confederation



A derivation

- Countries and provinces are regions on the island.
- Regions are collections of locations.
- xCy denotes the property of two regions to share at least one common border.

1. $aCa^* \ (= b + c + d + e + f + g)$.
2. $a(-C)d \ \Rightarrow \ aC(b + c + e + f + g)$.
3. $d = f + g \ \overset{2)}{\Rightarrow} aC(b + c + e)$.
4. c is strictly inside $c + e$, i.e. every country which does not have a border with e cannot share a border with c.
5. $e(-C)a \ \overset{3)}{\Rightarrow} aCb$.



Properties of the example

- Basic entities are regions.
- Spatial information is given with respect to other regions. No information about locations (points) is assumed.
- The information given is incomplete, so a derivation mechanism is needed.

What is space made of?

- The basic entity of Euclidean geometry are points.
- Points are abstract entities and do not exist in the physical world. How do points relate to everyday objects in space?
- Reasoning about regions (= sets of points) requires 2^{nd} order logic.
- Alternative: Choose regions as basic entity instead of points and define points (if at all) via sets of regions, see e.g. Biacino and Gerla [4], but also Schoop [31].

Points vs aggregates

A.N. Whitehead: The Organization of Thought, 1917

Our space concepts are concepts of relations between things in space. Thus there is no such entity as a self-subsistent point. A point is merely the name for some peculiarity of the relations between the matter which is, in common language, said to be in space.

It follows from the relative theory that a point should be definable in terms of the relations between material things. So far as I am aware, this outcome of the theory has escaped the notice of mathematicians, who have invariably assumed the point as the ultimate starting ground of their reasoning. Many years ago I explained some types of ways in which we might achieve such a definition, and more recently have added some others. Similar explanations apply to time. Before the theories of space and time have been carried to a satisfactory conclusion on the relational basis, a long and careful scrutiny of the definitions of points of space and instants of time will have to be undertaken, and many ways of effecting these definitions will have to be tried and compared. This is an unwritten chapter of mathematics, in much the same state as

Qualitative spatial reasoning - Mereotopology

- Investigates properties of relations "part–of" (P) and "contact"




- "Foundations of the General Theory of Sets" (Leśniewski, 1916)
- "Point, line, and surface as sets of solids" (de Laguna, 1922)
- "Geometry in a sensible world", (Nicod, 1924)
- "Foundation of the geometry of solids" (Tarski, 1929)
- "Process and reality" (Whitehead, 1929)
- "Axiomatization of Geometry without Points" (Grzegorczyk, 1960)

Modern approaches

- A calculus of individuals based on 'connection' (Clarke [6, 7], but see Biacino and Gerla [3])
- Computing Transitivity Tables: A Challenge for Automated Theorem Provers (Randell et al. [30])
- Parts, wholes, and part–whole relations: The prospect of mereotopology (Varzi [36])
- The mereotopology of discrete space (Galton [17])
- A note on proximity spaces and connection based mereology (Vakarelov et al. [35])
- Pointless Geometries (Gerla [18])
- Handbook of Spatial Logics (Aiello et al. [1])
- Qualitative Spatial and Temporal Reasoning (Ligozat [25])

Today's applications

- Geographical information systems.
- Computer games.
- Semantic web (Ontologies are regions).
- Biological systems.
- Mobile robot navigation.
- Computer aided design
- more

See Wolter and Wallgrün [38].

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> *"Spatial databases will benefit from the composition table of topological relations if it is applied during data acquisition to integrate independently collected topological information and to derive new topological knowledge ; to detect consistency violations among spatial data about some otherwise non–evident topological facts; or during query processing, when spatial queries are less expensive to be executed or involve less objects." (Egenhofer [16])*

Binary relations

- A binary relation on U is a subset of $U \times U$, $Rel(U) = 2^{U \times U}$.
- Convention: $\langle x, y \rangle \in R \iff xRy$.
- Special relations: $\emptyset$, V, $1'$, $0'$, where

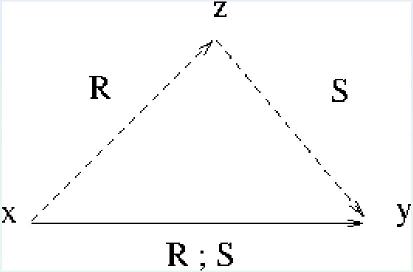
$$\begin{aligned} V &= U \times U, \\ 1' &= \{\langle x, x \rangle : x \in U\}, \\ 0' &= \{\langle x, y \rangle : x, y \in U, x \neq y\} = V \setminus 1'. \end{aligned}$$

- $\langle Rel(U), \cap, \cup, -, \emptyset, V \rangle$ is a Boolean algebra.

Relative operations

Composition (relative multiplication):

$$R \; ; \; S = \{\langle x, y\rangle : (\exists z)[xRz \text{ and } zSy]\}$$



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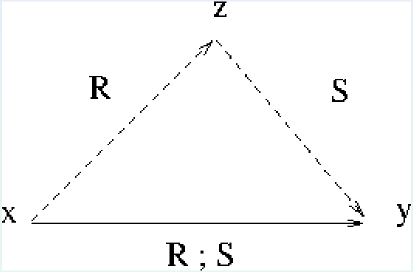

Converse:

$$R^{\smile} = \{\langle y,x\rangle : xRy\}.$$

Properties of the operators [5, 22]

1. $;$ is associative, i.e. $(R \; ; \; S) \; ; \; T = R \; ; \; (S \; ; \; T)$.
2. $1' \; ; \; R = R = R \; ; \; 1'$.
   ➦ $\langle Rel(U), \; ; \; , 1' \rangle$ is a monoid.

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3. $R^{\smile\smile} = R$.
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$$R \diagdown_{\mathrm{res}} S = -(R^{\smile} \; ; \; -S),$$
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6. $(R \,;\, S) \cap T = \emptyset \iff (T \,;\, S^{\smile}) \cap R = \emptyset \iff (R^{\smile} \,;\, T) \cap S = \emptyset$.

Algebras of binary relations

- $\langle Rel(U), \cap, \cup, -, \emptyset, V, ;, \breve{}, 1' \rangle$ is called the *full algebra of binary relations on U*.
- If $A \subseteq Rel(U)$ is closed under the operations and contains the distinguished constants $\emptyset, V, 1'$, it is called an *algebra of binary relations* (BRA).

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- The equational theory of (B)RAs can express inequality:

$$\tau \neq \sigma \Longleftrightarrow \tau \oplus \sigma \neq \emptyset \Longleftrightarrow V \; ; \; (\tau \oplus \sigma) \; ; \; V = V.$$

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➽ BRAs are not locally finite.

Composition tables

- A finite BRA is a complete atomic Boolean algebra, and the action of the Boolean operators are uniquely determined by the atoms.
- Since ; and ˘ distribute over $\cup$ it suffices to specify composition and converse of atoms.

;	S	$R\ ;\ S = T_0 \cup T_1 \cup \ldots \cup T_k$
R	$T_0, T_1, \ldots, T_k$	

- Atoms below $1'$ need not be listed.

Weak composition

- Splitting equality leads to two kinds of theorem:

$$(\forall x, y, z)[xRz \wedge zSy \Rightarrow xT_0y \vee \cdots \vee xT_ky],$$
$$(\forall x, y)[xT_iy \Rightarrow (\exists z)xRz \wedge zSy].$$

- Considering only the first direction leads to *weak composition*:

$;_w$	S	$R\ ;\ S \subseteq T_0 \cup T_1 \cup \ldots \cup T_k$
R	$T_0, T_1, \ldots, T_k$	

- An interpretation of a weak composition table is *extensional* or *path consistent* if $;_w = ;$.
- A table - regarded as an abstract structure - can have extensional and non–extensional interpretations.

Contact structures

A contact structure is a triple $\langle U, P, \mathscr{C} \rangle$ where U is a set (of regions), P a partial order on U, and $\mathscr{C}$ a binary relation ("contact") which satisfies

1. $\mathscr{C}$ is symmetric, i.e. $a \mathscr{C} b$ implies $b \mathscr{C} a$ for all $a, b \in U$.
2. $\mathscr{C}$ is reflexive, i.e. $a \mathscr{C} a$ for all $a \in U$.
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$$2. \Rightarrow x \mathscr{C} x \stackrel{3.}{\Rightarrow} x \mathscr{C} y.$$

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➥ $\mathscr{C}$ is extensional iff $\mathscr{C}(x) = \mathscr{C}(y) \iff x = y$ iff $P = \mathscr{C} \diagdown_{\text{res}} \mathscr{C}$.

Mereological relations

$PP = P \setminus 1'$,	proper part
$O = P^{\smile} \; ; \; P$	overlap
$PO = O \setminus (P \cup P^{\smile})$	partial overlap
$DR = (U \times U) \setminus O$	disjoint

Mereological relations

$$\begin{aligned} PP &= P \setminus 1', && \text{proper part} \\ O &= P^{\smile} \ ; \ P && \text{overlap} \\ PO &= O \setminus (P \cup P^{\smile}) && \text{partial overlap} \\ DR &= (U \times U) \setminus O && \text{disjoint} \end{aligned}$$

Note: $O \subseteq \mathscr{C}$:

$$xOy \Rightarrow (\exists z)[xP^{\smile} zPy] \Rightarrow (\exists z)[x\mathscr{C} zPy] \Rightarrow x(\mathscr{C} \ ; \ P)y \Rightarrow x\mathscr{C} y.$$

RCC5 relations:

$$U \times U = 1' \quad \dot{\cup} \quad PP \quad \dot{\cup} \quad PP^{\smile} \quad \dot{\cup} \quad PO \quad \dot{\cup} \quad DR.$$

RCC5 relations on open disks

Let U be the collection of all open disks in the plane,and

$$xPy \iff x \subseteq y.$$

The BRA generated by P on $Rel(U)$ has five atoms and the table

$;$	PP	$PP^{\smile}$	PO	DR
PP	PP	V	PP, PO, DR	DR
$PP^{\smile}$	$-DR$	$PP^{\smile}$	$PP^{\smile}, PO$	$PP^{\smile}, PO, DR$
PO	PP, PO	$PP^{\smile}, PO, DR$	V	$PP^{\smile}, PO, DR$
DR	PP, PO, DR	DR	PP, PO, DR	V

$\mathscr{C} = P \cup P^{\smile} \cup PO$ is extensional!.

RCC8 relations

If $\mathscr{C} \neq O$, PP and $\mathscr{C}$ are split:

$$EC \stackrel{\text{def}}{=} \mathscr{C} \cap -O \qquad \text{external contact} \qquad (1)$$

$$TPP \stackrel{\text{def}}{=} PP \cap (EC \; ; \; EC) \qquad \text{tangential proper part} \qquad (2)$$

$$NTPP \stackrel{\text{def}}{=} PP \cap -TPP \qquad \text{non–tangential proper part} \qquad (3)$$

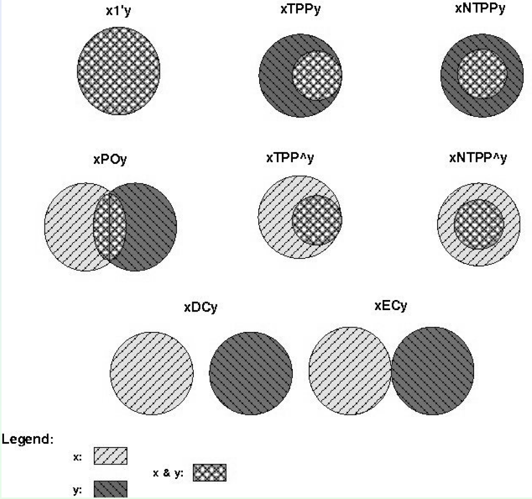
$$DC \stackrel{\text{def}}{=} -\mathscr{C} \qquad \text{disconnected} \qquad (4)$$

The RCC8 relations

$$1', TPP, TPP^{\smile}, NTPP, NTPP^{\smile}, PO, EC, DC$$

partition $U \times U$.
These are the two–dimensional version of Allen's interval relations [2], see [11] for details.

RCC8 relations on closed disks (RCC8, [30])



Closed disk composition table

;		C					
	DR		O				
				PP		PP˘	
	DC	EC	PO	TPP	NTPP	TPP˘	NTPP˘
DC	V	DR,PO,PP	DR,PO,PP	DR,PO,PP	DR,PO,PP	DC	DC
EC	DR,PO,PP˘	1',DR,PO, TPP TPP˘	DR,PO,PP	EC,PO,PP	PO,PP	DR	DC
PO	DR,PO,PP˘	DR,PO,PP˘	V	PO,PP	PO,PP	DR,PO,PP˘	DR,PO,PP˘
TPP	DC	DR	DR,PO,PP	PP	NTPP	1',DR,PO, TPP,TPP˘	DR,PO, PP˘
NTPP	DC	DC	DR,PO,PP	NTPP	NTPP	DR,PO,PP	1
TPP˘	DR,PO,PP˘	EC,PO,PP˘	PO,PP˘	1',PO, TPP,TPP˘	PO,PP	PP˘	NTPP˘
NTPP˘	DR,PO,PP˘	PO,PP˘	PO,PP˘	PO,PP˘	O	NTPP˘	NTPP˘

- Also known as the *RCC8 composition table* (a misnomer).
- $\mathscr{C}$ is extensional.

Empiricism and rationalism (Pratt-Hartmann [29])

The empiricist:

1. Select a group of primitive spatial relations corresponding to familiar spatial concepts and illustrate their meaning with a few examples.
2. Write down axioms to govern these primitives.
3. Propose various definitions for a range of familiar spatial relations not included in the primitive ones.

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The rationalist:

1. Select a group σ of predicate letters to represent primitive spatial relations corresponding to familiar spatial concepts.
2. Using some familiar point-based model of space, select a set A of subsets of that space to count as regions recognized by the theory.
3. Interpret the symbols from σ over the regions in A using the standard definitions to obtain a structure $A(\sigma)$.
4. Systematically investigate its properties as a structure.

Brief review of topological terms

- A *topology* over a set X is a collection τ of subsets of X such that
  1. $\emptyset, X \in \tau$,
  2. $O_1, O_2 \in \tau$ implies $O_1 \cap O_2 \in \tau$,
  3. $\{O_i : i \in I\} \subseteq \tau$ implies $\bigcup\{O_i : i \in I\} \in \tau$
- Elements of τ are called *open sets* and their complements *closed sets*.
- For $Y \subseteq X$, $\mathrm{int}(Y)$ is the largest open set contained in Y, and $\mathrm{cl}(Y)$ the smallest closed sets containing Y.
- The boundary of Y is the set $\mathrm{bd}(Y) = \mathrm{cl}(Y) \setminus \mathrm{int}(Y)$.
- An open basis for τ is a subset $\mathscr{B}$ of τ such that each $O \in \tau$ is a union of elements of $\mathscr{B}$.
- A closed basis for τ is a set $\mathscr{B}$ of closed sets 2^X such that every closed set is an intersection of elements of $\mathscr{B}$.

Separation axioms

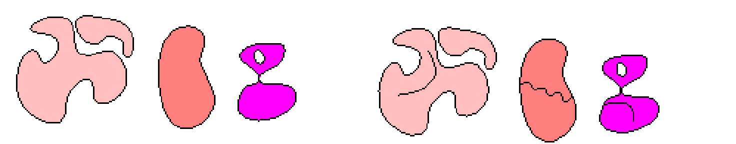
Let $\mathscr{X} = \langle X, \tau \rangle$ be a topological space. $\mathscr{X}$ is called a

1. T_0 *space* if for all $x, y \in \tau$ there is some $O \in \tau$ such that $x \in O$ and $y \notin O$ or $x \notin O$ and $y \in O$.
2. T_1 *space* if for all $x, y \in X$ there is some $O \in \tau$ $x \in O$ and $y \notin O$.
3. T_2 *space* if for all $x, y \in X, x \neq y$ there are $O_1, O_2 \in \tau$ such that $x \in O_1, y \in O_2$ and $O_1 \cap O_2 = \emptyset$.
4. *regular space*, if for every $x \in X$ and every closed set A with $x \notin A$ there are open sets O_1, O_2 such that $x \in O_1, A \subseteq O_2$ and $O_1 \cap O_2 = \emptyset$.
5. *weakly regular space*, if for every $x \in X$ and every regular closed set A with $x \notin A$ there are open sets O_1, O_2 such that $x \in O_1, A \subseteq O_2$ and $O_1 \cap O_2 = \emptyset$.

Regular sets and more

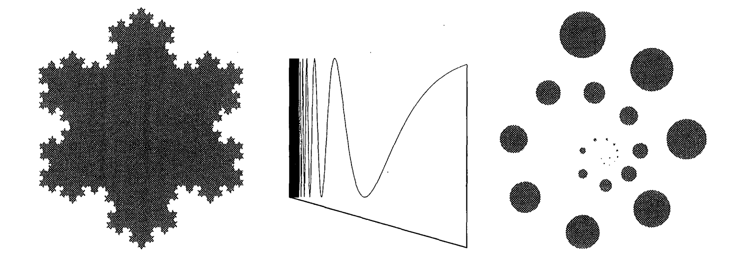
- $Y \subseteq X$ is *regular open* if $\mathrm{int}(\mathrm{cl}(Y)) = Y$, and *regular closed* if $\mathrm{cl}(\mathrm{int}(Y)) = Y$.

Figure: Regular and nonregular sets (from Pratt and Schoop [27])



- A topology τ is called *semiregular* if it has a basis of regular open sets.
- A topology τ is called *connected* if the only closed–open sets are $\emptyset$ and X.
- A topology τ is called *totally disconnected* if every open set is the union of closed–open sets.

Pathological regular sets (from Pratt and Schoop [28])



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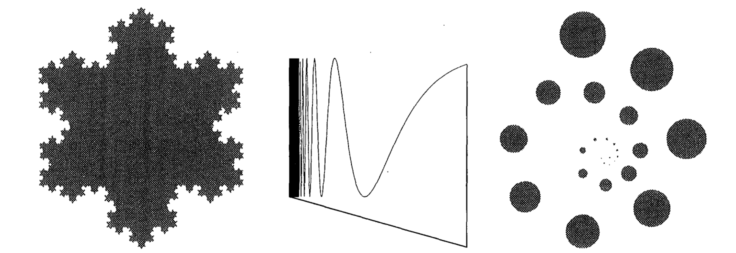
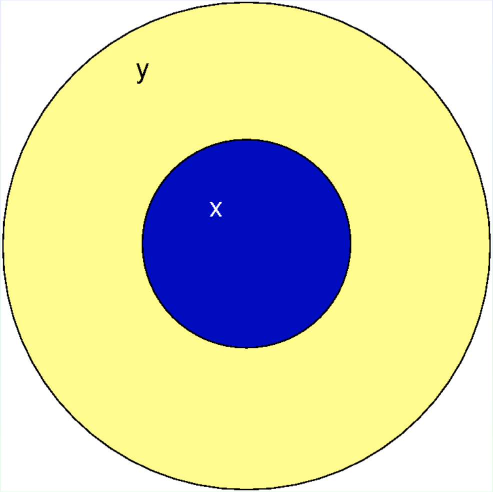


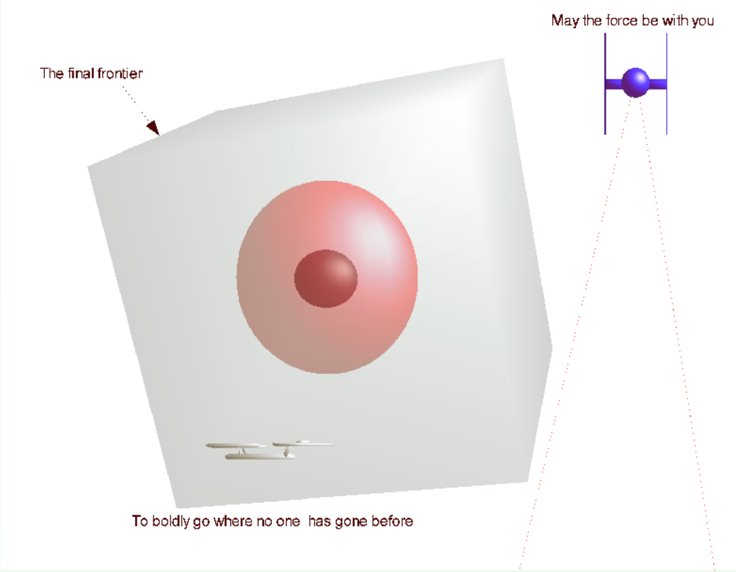
- One may exclude pathological regions by considering polygonal or semi–algebraic sets as regions [27, 28, 31].

A hole in the plane

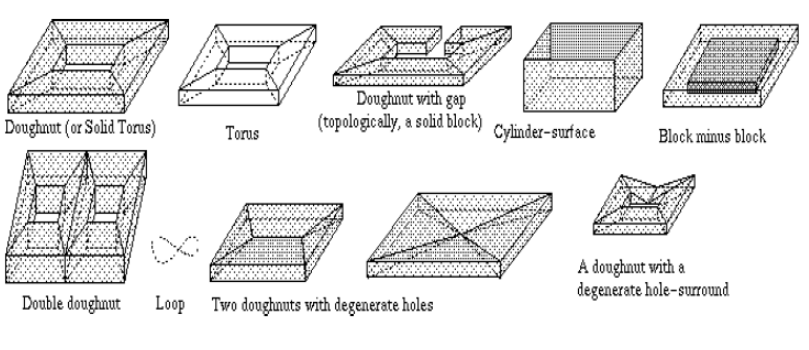
xHy iff $xECy$ and $(\forall z)[zECx \Rightarrow zOy]$.



A hole in space



Gotts' doughnuts [19, 20]

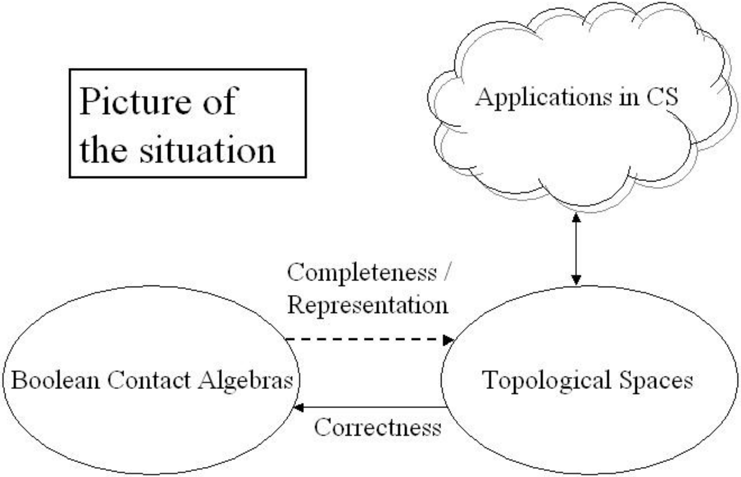


Our setup

- *Concrete* : Regions in a topological space and operations and relations among them.
- *Abstract* : A contact structure $\langle U, P, C \rangle$ and an algebraic structure on U which is in some sense compatible with C.
- *Bridge* : Sound and complete axiomatizations and representation theorems.

Be parsimonious!

Picture of the situation



Concrete structures – Regular closed sets

- Regions are regular closed sets of some semiregular topological space.
- The regular closed sets of a topological space X form a complete Boolean algebra under the operations

$$
\begin{aligned}
a + b &= a \cup b, \\
a \cdot b &= \mathrm{cl}(\mathrm{int}(a \cap b)), \\
-a &= \mathrm{cl}(X \setminus a),\ 0 = \emptyset,\ 1 = X.
\end{aligned}
$$

  Observe that it is possible that $a \cdot b = 0$, but $a \cap b \neq \emptyset$.
- A *standard contact structure* is a subalgebra B of the Boolean algebra $\mathrm{RegCl}(X)$ of regular closed sets of a (semiregular) topological space $\langle X, \tau \rangle$, enhanced by a contact relation C_τ such that for all regular closed sets $a, b \in B$

$$
a \mathscr{C}_\tau b \iff a \cap b \neq \emptyset.
$$

  The part-of relation P is set inclusion.

Abstract structures – Boolean contact algebras

A *Boolean contact algebra* $\langle B, \leq, \mathscr{C} \rangle$ is a Boolean algebra B together with its ordering $\leq$ and a binary relation $\mathscr{C}$ on B which satisfies for all $x, y, z \in B$

C_0. $0(-\mathscr{C})x$

C_1. $x \neq 0$ implies $x\mathscr{C}x$ (domain reflexivity)

C_2. $x\mathscr{C}y$ implies $y\mathscr{C}x$ (symmetry)

C_3. $x\mathscr{C}y$ and $y \leq z$ implies $x\mathscr{C}z$. (compatibility

C_4. $x\mathscr{C}(y+z)$ implies ($x\mathscr{C}y$ or $x\mathscr{C}z$) (distributivity)

B is connected if

- $x \neq 0$ and $x \neq 1$ implies $x\mathscr{C} - x$ (connectivity).

B is extensional if

- If $x\mathscr{C}z \iff y\mathscr{C}z$ for all $z \in B$, then $x = y$ (extensionality) (Whitehead!).

Recall : If $\mathscr{C}$ is extensional, then $\leq$ can be defined by $\mathscr{C}$!

Basic facts on BCAs

Let B be a Boolean algebra.

- There is exactly one contact relation $\mathscr{C}$ if $B = \{0,1\}$, namely, $\mathscr{C} = \{\langle 1,1 \rangle\}$.

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- The smallest contact relation on B is given by

$$C_{\text{min}} = \{\langle x,y\rangle : x \cdot y \neq 0\}.$$

$\mathscr{C}_{\text{min}}$ is the *overlap relation* , usually denoted by O.

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$$\mathscr{C}_{\max} = \{\langle x,y\rangle : x,y \neq 0\}.$$

  $\mathscr{C}_{\max}$ is connected but (usually) not extensional.
- The class of finite Boolean contact algebras has the joint embedding property and the amalgamation property (Düntsch and Li [12]).

Atomless BCAs

- If $\mathscr{C}$ is connected and extensional, then B is atomless [13]. Preparation: $\mathscr{C}$ is extensional if and only for all $a \neq 0, 1$ there is some $b \neq 0$ such that $a(-\mathscr{C})b$.

Proof.

Assume a is an atom of B. Since $\mathscr{C}$ is connected, we have $a\mathscr{C} - a$. Now, $-a$ is an antiatom, so, if $b \neq 0, a$, then $b \cdot -a \neq 0$. Hence, $-a$ is in contact with all nonzero elements of B, contradicting that $\mathscr{C}$ is extensional. □

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- If B is an atomless BA, then there is an extensional and connected contact relation on B [23].

Proof.

Exercise. □

A simple construction of contact algebras (Düntsch and Winter [15], Koppelberg et al. [23]

Let B be a subalgebra of $\mathscr{P}(X)$, and R a symmetric and reflexive relation on X. R induces a contact relation $\mathscr{C}$ on B by

$$b \mathscr{C} c \iff (\exists x, y \in X)[x \in b \text{ and } y \in c \text{ and } xRy],$$

i.e.

$$\iff (b \times c) \cap R \neq \emptyset.$$

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If B is a BCA, the relation R on $\mathrm{Ult}(B)$ defined by

$$xRy \iff x \times y \subseteq \mathscr{C}$$

is symmetric, reflexive and closed in the product topology of $\mathrm{Ult}(B)$.

From concrete to abstract

Let $\mathscr{X} = \langle X, \tau \rangle$ be a semiregular space, and $\mathrm{RegCl}(\mathscr{X})$ be the BA of regular closed sets with standard connection $\mathscr{C}$, i.e.

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- $\mathrm{RegCl}(\mathscr{X})$ is a Boolean contact algebra (see Biacino and Gerla [4]).

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Proof.
"$\Rightarrow$": Assume there are disjoint nonempty open sets a, b whose union is X. Then, a, b are regular closed and $b = -a$. Since $\mathscr{C}$ is connected, $a \mathscr{C} b$, i.e. $a \cap b \neq \emptyset$, a contradiction.
"$\Leftarrow$": Let $a \neq \emptyset, X$ and $a(-\mathscr{C}) - a$. Thus, $a \cap \text{cl}(X \setminus a) = \emptyset$ and $a \cup \text{cl}(X \setminus a) = X$, showing that τ is not connected.

□

From abstract to concrete

Famous representation results:

- Each finite group is isomorphic to a group of permutations.
- Each Boolean algebra is isomorphic to an algebra of sets.

From abstract to concrete

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- Each finite group is isomorphic to a group of permutations.
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Task : Given a BCA $\langle B, \leq, \mathscr{C} \rangle$, find a topological space $\langle X, \tau \rangle$ with standard contact $\mathscr{C}_\tau$ and an embedding from $\langle B, \leq, \mathscr{C} \rangle$ into $\langle \mathrm{RegCl}(X), \subseteq, \mathscr{C}_\tau \rangle$.

Preliminary definitions : A *clan* is a nonempty subset Γ of B such that

1. If $a \in \Gamma$ and $a \leq b$, then $b \in \Gamma$.
2. If $a, b \in \Gamma$, then $a \mathscr{C} b$.
3. If $a + b \in \Gamma$, then $a \in \Gamma$ or $b \in \Gamma$.

Representation Theorem([9, 14])

Let X be the set of all clans on B and define $h : B \to 2^X$ by

$$h(a) = \{\Gamma \in X : a \in \Gamma\}.$$

$\mathscr{B} = \{h(a) : a \in B\}$ is closed under union:

$$\begin{aligned} h(a) \cup h(b) &= \{\Gamma \in X : a \in \Gamma\} \cup \{\Gamma \in X : b \in \Gamma\} \\ &= \{\Gamma \in X : a \in \Gamma \text{ or } b \in \Gamma\} \\ &= \{\Gamma \in X : a + b \in \Gamma\} \\ &= h(a + b) \end{aligned}$$

Let τ be the topology the basis $\{X \setminus h(a) : a \in B\}$. Then,

- Each $h(a)$ is regular closed.
- The mapping $h : B \to \mathrm{RegCl}(X)$ is injective and preserves the Boolean operations.
- $a \mathscr{C} b$ if and only if $h(a) \mathscr{C}_\tau h(b)$.

A representation theorem for Boolean algebras

0. *On each atomless Boolean algebra there is a connected and extensional contact relation* (Koppelberg et al. [23]).

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2. *For every connected and extensional Boolean contact algebra there is a connected compact weakly regular T_1 space X such that B is isomorphic to a subalgebra of* $\langle \mathrm{RegCl}(X), \mathscr{C}_w \rangle$ (Dimov and Vakarelov [10], Düntsch and Winter [14]).

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Děkuji
Thank you
Dziękuję
Danke
Merci

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