Eduard Bartl: Minimal Solutions of Fuzzy Relational Eq.: Probabilistic Algorithm

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Introduction

- generalized fuzzy relational equations (FREs) are equations of the form $R \boxdot S = T$, where R, S, and T are binary fuzzy relations, and $\boxdot$ denotes a general composition introduced by Radim Belohlavek in [3]
- problem: given two of the relations, determine the third one for which the equality $R \boxdot S = T$ holds
- the aim of the current research: extension of the previous work on generalized FREs by developing a method for constructing all minimal solutions and, consequently, for determining the whole solution set

Preliminaries

- an aggregation structure $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \square\rangle$, where $\mathbf{L}_i$ are complete lattices and $\square : L_1 \times L_2 \to L_3$ is a function which commutes with suprema in both arguments; define operations:

$$\circ_{\square} : L_1 \times L_3 \to L_2, \quad a_1 \circ_{\square} a_3 = \textstyle\bigvee_2 \{a_2 \,|\, a_1 \square a_2 \leq_3 a_3\}$$
$${}_{\square}\!\circ : L_3 \times L_2 \to L_1, \quad a_3 \,{}_{\square}\!\circ\, a_2 = \textstyle\bigvee_1 \{a_1 \,|\, a_1 \square a_2 \leq_3 a_3\}$$
$${}^{\mathrm{op}}_{\square}\!\circ : L_3 \times L_2 \to L_1, \quad a_3 \,{}^{\mathrm{op}}_{\square}\!\circ\, a_2 = \textstyle\bigwedge_1 \{a_1 \,|\, a_1 \square a_2 \geq_3 a_3\}$$

- consider two important examples of aggreg. struct.; in both cases, $\langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ is a complete residuated lattice; $L_i = L$ and $\leq_i$ is either $\leq$ or the dual of $\leq$

1. $\mathbf{L}_1 = \langle L, \leq\rangle$, $\mathbf{L}_2 = \langle L, \leq\rangle$, $\mathbf{L}_3 = \langle L, \leq\rangle$, $\square = \otimes$:

$$a_1 \circ_{\square} a_3 = \bigvee\{a_2 \,|\, a_1 \otimes a_2 \leq a_3\} = a_1 \to a_3$$
$$a_3 \,{}_{\square}\!\circ\, a_2 = \bigvee\{a_1 \,|\, a_1 \otimes a_2 \leq a_3\} = a_2 \to a_3$$

2. $\mathbf{L}_1 = \langle L, \leq\rangle$, $\mathbf{L}_2 = \langle L, \leq^{-1}\rangle$, $\mathbf{L}_3 = \langle L, \leq^{-1}\rangle$, $\square = \to$:

$$a_1 \circ_{\square} a_3 = \bigwedge\{a_2 \,|\, a_1 \to a_2 \geq a_3\} = a_1 \otimes a_3$$
$$a_3 \,{}_{\square}\!\circ\, a_2 = \bigvee\{a_1 \,|\, a_1 \to a_2 \geq a_3\} = a_3 \to a_2$$

- for fuzzy relations $R \in L_1^{X\times Y}$, $S \in L_2^{Y\times Z}$, let a fuzzy relation $R \boxdot S \in L_3^{X\times Z}$ be defined by

$$(R \boxdot S)(x, z) = \textstyle\bigvee_{3\, y\in Y} (R(x, y) \square S(y, z))$$

- product $\boxdot$ generalizes both sup-t-norm product ($\circ$) and inf-residuum product ($\triangleleft$):
  for the setting of Example 1: $R \boxdot S = R \circ S$
  for the setting of Example 2: $R \boxdot S = R \triangleleft S$
- for $R \in L_1^{X\times Y}$ and $S \in L_3^{Y\times Z}$, let $R \triangleleft_{\square} S \in L_2^{X\times Z}$ and $R \,{}_{\square}\!\triangleleft\, S \in L_1^{X\times Z}$ be defined by

$$(R \triangleleft_{\square} S)(x, z) = \textstyle\bigwedge_{2\, y\in Y} (R(x, y) \circ_{\square} S(y, z))$$
$$(R \,{}_{\square}\!\triangleleft\, S)(x, z) = \textstyle\bigwedge_{1\, y\in Y} (R(x, y) \,{}_{\square}\!\circ\, S(y, z))$$

Minimal Solutions

- assumption: $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \square\rangle$ be an aggreg. struct. such that $\square$ commutes with infima in the first arg.
- scalar-by-scalar equation is expression

$$u \square s = t, \qquad (1)$$

where $u \in L_1$, $s \in L_2$, $t \in L_3$

Theorem 1 If equation (1) is solvable then for each solution $r \in L_1$ we have $r \in [t \,{}^{\mathrm{op}}_{\square}\!\circ\, s, t \,{}_{\square}\!\circ\, s]$.

- vector-by-vector equation is expression

$$\begin{pmatrix} u_1 \ldots u_n \end{pmatrix} \boxdot \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} = t, \qquad (2)$$

where $(u_j) \in L_1^Y$, $(s_j)^{-1} \in L_2^Y$, $t \in L_3$, $j \in J$

Theorem 2 If there is $j' \in J$ such that $u_{j'} \square s_{j'} = t$ is solvable then equation (2) has a minimal solution $R = (r_1 \ldots r_n)$ such that

$$r_j = \begin{cases} t \,{}^{\mathrm{op}}_{\square}\!\circ\, s_j & \text{for } j = j', \\ 0_1 & \text{otherwise.} \end{cases}$$

- vector-by-matrix equation is expression

$$\begin{pmatrix} u_1 \ldots u_n \end{pmatrix} \boxdot \begin{pmatrix} s_{11} \ldots s_{1p} \\ \vdots \ \ddots \ \vdots \\ s_{n1} \ldots s_{np} \end{pmatrix} = \begin{pmatrix} t_1 \ldots t_p \end{pmatrix}, \qquad (3)$$

where $(u_j) \in L_1^Y$, $(s_{jk}) \in L_2^{Y\times Z}$, $(t_k) \in L_3^Z$, $j \in J$, $k \in K$

- equation (3) can be rewritten using the table $\mathfrak{T}$:

$$\mathfrak{T}_{jk} = \begin{cases} u_j \square s_{jk} & \text{for } j \in J \\ t_k & \text{for } j = n+1 \end{cases}$$

- now, we need to suppose $\mathbf{L}_1$ is a chain and ${}_{\square}\!\circ = {}^{\mathrm{op}}_{\square}\!\circ$
- important result: when we put the greatest solution into the equation (3), the k-th column of $\mathfrak{T}$ consists of the values equal to t_k or the values that are strictly smaller than t_k; this allows us to binarize table $\mathfrak{T}$: $\mathfrak{B}_{jk} = 1$ if $\mathfrak{T}_{jk} = t_k$, $\mathfrak{B}_{jk} = 0$ if $\mathfrak{T}_{jk} <_3 t_k$
- we say that $J_{\mathrm{cov}} \subseteq J$ is a covering of the last row of $\mathfrak{B}$ if $\max_{j\in J_{\mathrm{cov}}} \mathfrak{B}_{jk} = 1$ for all $k \in K$; we say that $J_{\mathrm{cov}} \subseteq J$ is a minimal covering of the last row of $\mathfrak{B}$ if there is no covering J'_{cov} such that $J'_{\mathrm{cov}} \subset J_{\mathrm{cov}}$

Theorem 3 Let $\hat{R} = (\hat{r}_1 \ldots \hat{r}_n)$ be the greatest solution of a solvable equation (3). Then every minimal solution $\check{R} = (\check{r}_1 \ldots \check{r}_n)$ of (3) is of the form:

$$\check{r}_j = \begin{cases} \hat{r}_j & \text{for } j \in J_{\mathrm{cov}}, \\ 0_1 & \text{otherwise,} \end{cases}$$

where J_{cov} is a minimal covering of the last row of $\mathfrak{B}$.

Algorithm

- algorithm requires the table $\mathfrak{B}$ and returns a family of (almost) minimal coverings of the last row in $\mathfrak{B}$
- let $J_{\mathrm{cov}} \subseteq J$ denote a set of row-indices in table $\mathfrak{B}$ representing an (almost) minimal covering, let $K_{\mathrm{unc}} \subseteq K$ denote a set of uncovered column-indices of the last row in table $\mathfrak{B}$, and let $\mathfrak{B}_{j\_} \subseteq K$ denote a set of column-indices representing the j-th row in table $\mathfrak{B}$, i.e.

$$k \in \mathfrak{B}_{j\_} \text{ iff } \mathfrak{B}_{jk} = 1$$

for all $k \in K$

- the crucial part of algorithm is calculating a minimal covering: in a given step of computation, the problem of finding a minimal covering can be identified with the set cover problem
- since the set cover problem is NP-hard optimization problem, we apply the greedy approach
- such approach guarantees a known approximation ratio of the algorithm (note, the approximation ratio tells us how far the obtained solution is from a minimal solution in the worst case)

Probabilistic algorithm for computing all minimal solutions based on greedy approach:

Require: binary table $\mathfrak{B}$ of dimension $(n+1) \times p$, last row of $\mathfrak{B}$ is full of 1s
Ensure: family $\mathcal{C}$ of (almost) minimal coverings of the last row in table $\mathfrak{B}$

$\mathcal{C} \leftarrow \emptyset$
$J \leftarrow \{1, 2, \ldots, n\}$
repeat
  $J_{\mathrm{cov}} \leftarrow \emptyset$
  $K_{\mathrm{unc}} \leftarrow \{1, 2, \ldots, p\}$
  while ($K_{\mathrm{unc}} \neq \emptyset$) and ($J_{\mathrm{cov}} \neq J$) **do**
    select $j \in J$ for which $|\mathfrak{B}_{j\_} \cap K_{\mathrm{unc}}|$ is largest
    $J_{\mathrm{cov}} \leftarrow J_{\mathrm{cov}} \cup \{j\}$
    $K_{\mathrm{unc}} \leftarrow K_{\mathrm{unc}} \setminus \mathfrak{B}_{j\_}$
  end while
  if $K_{\mathrm{unc}} = \emptyset$ **then**
    $\mathcal{C} \leftarrow \mathcal{C} \cup \{J_{\mathrm{cov}}\}$
    randomly choose $j \in J_{\mathrm{cov}}$
    remove j-th row from table $\mathfrak{B}$
    $J \leftarrow J \setminus \{j\}$
  end if
until $K_{\mathrm{unc}} \neq \emptyset$
return $\mathcal{C}$

Example

- we assume a five-element Gödel chain and a vector-by-matrix sup-t-norm eq. $U \circ S = T$, where

$$S = \begin{pmatrix} 0.50 & 0.50 & 0.25 & 0.50 \\ 1.00 & 1.00 & 0.00 & 0.25 \\ 1.00 & 0.50 & 0.00 & 0.00 \\ 0.25 & 0.75 & 0.00 & 0.25 \\ 0.75 & 0.00 & 0.75 & 0.75 \\ 0.25 & 0.50 & 0.00 & 1.00 \end{pmatrix}, \quad T = \begin{pmatrix} 0.75 & 0.75 & 0.25 & 0.50 \end{pmatrix}$$

- the equation is solvable with the greatest solution being

$$\hat{R} = \begin{pmatrix} 1.00 & 0.75 & 0.75 & 1.00 & 0.25 & 0.50 \end{pmatrix}$$

- tables $\mathfrak{T}$ and $\mathfrak{B}$, respectively:

$r_1 \otimes s_{11} = 0.50$	$r_1 \otimes s_{12} = 0.50$	$r_1 \otimes s_{13} = \mathbf{0.25}$	$r_1 \otimes s_{14} = \mathbf{0.50}$
$r_2 \otimes s_{21} = \mathbf{0.75}$	$r_2 \otimes s_{22} = \mathbf{0.75}$	$r_2 \otimes s_{23} = 0.00$	$r_2 \otimes s_{24} = 0.25$
$r_3 \otimes s_{31} = \mathbf{0.75}$	$r_3 \otimes s_{32} = 0.50$	$r_3 \otimes s_{33} = 0.00$	$r_3 \otimes s_{34} = 0.00$
$r_4 \otimes s_{41} = 0.25$	$r_4 \otimes s_{42} = \mathbf{0.75}$	$r_4 \otimes s_{43} = 0.00$	$r_4 \otimes s_{44} = 0.25$
$r_5 \otimes s_{51} = 0.25$	$r_5 \otimes s_{52} = 0.00$	$r_5 \otimes s_{53} = \mathbf{0.25}$	$r_5 \otimes s_{54} = 0.25$
$r_6 \otimes s_{61} = 0.25$	$r_6 \otimes s_{62} = 0.50$	$r_6 \otimes s_{63} = 0.00$	$r_6 \otimes s_{64} = \mathbf{0.50}$
$t_1 = \mathbf{0.75}$	$t_2 = \mathbf{0.75}$	$t_3 = \mathbf{0.25}$	$t_4 = \mathbf{0.50}$

0	0	1	1
1	1	0	0
1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1
1	1	1	1

- minimal coverings: $\{1, 2\}$, $\{1, 3, 4\}$, $\{2, 5, 6\}$, $\{3, 4, 5, 6\}$
- minimal solutions of $U \circ S = T$:

$$\check{R}_1 = (\mathbf{1.00}\ \ \mathbf{0.75}\ \ 0.00\ \ 0.00\ \ 0.00\ \ 0.00)$$
$$\check{R}_2 = (\mathbf{1.00}\ \ 0.00\ \ \mathbf{0.75}\ \ \mathbf{1.00}\ \ 0.00\ \ 0.00)$$
$$\check{R}_3 = (0.00\ \ \mathbf{0.75}\ \ 0.00\ \ 0.00\ \ \mathbf{0.25}\ \ \mathbf{0.50})$$
$$\check{R}_4 = (0.00\ \ 0.00\ \ \mathbf{0.75}\ \ \mathbf{1.00}\ \ \mathbf{0.25}\ \ \mathbf{0.50})$$

Future Research

- complexity issues, efficient computation of all solutions (removing duplicities)

References

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