Miroslav Kolařík: Independence of the axiomatic system for MV-algebras

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ABSTRACT

We prove that the well-known axiomatic system of MV-algebras is not independent. The axiom of commutativity can be deleted and the remaining axioms are shown to be independent.

We describe an independent axiomatic system for MV-algebras. The concept of an MV-algebra was introduced by C. C. Chang [1] as an axiomatization of the Łukasiewicz many-valued logic. The definition used in nowadays is taken from the monograph [2] (with a different order of axioms):

DEFINITION

By an **MV-algebra** is meant an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following axioms

(M1) $x \oplus 0 = x$

(M2) $\neg\neg x = x$

(M3) $x \oplus y = y \oplus x$

(M4) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$

(M5) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

(M6) $x \oplus \neg 0 = \neg 0.$

We show that the axiomatic system (M1)–(M6) is redundant.

THEOREM

An algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ is an MV-algebra if and only if it satisfies the axioms (M1), (M2), (M4), (M5), (M6).

Proof

We need to show that (M3) follows from the axioms (M1), (M2), (M4), (M5) and (M6). For this, take $y = 0$ and substitute z by y in (M4) to obtain

$$(x \oplus 0) \oplus y = x \oplus (0 \oplus y).$$

Applying (M1), we get

(1) $x \oplus y = x \oplus (0 \oplus y)$.

Further, take $x = 0$ and substitute y by x in (M5) to compute

$$\neg(\neg 0 \oplus x) \oplus x = \neg(\neg x \oplus 0) \oplus 0$$

thus, applying (M1) and (M2), we obtain

(2) $\neg(\neg 0 \oplus x) \oplus x = x.$

For the next step, we start with (2) where instead of all x is $0 \oplus x$. Thus we have

$$0 \oplus x = \neg(\neg 0 \oplus (0 \oplus x)) \oplus (0 \oplus x).$$

The right hand side of the last identity can be reduced using (1) twice and (2) as follows

$$\neg(\neg 0 \oplus (0 \oplus x)) \oplus (0 \oplus x) = \neg(\neg 0 \oplus (0 \oplus x)) \oplus x = \neg(\neg 0 \oplus x) \oplus x = x.$$

Therefore, we have

(3) $0 \oplus x = x$.

Now, put $\neg(\neg x \oplus y)$ instead of x in (M4) to obtain

$$\neg(\neg x \oplus y) \oplus (y \oplus z) = (\neg(\neg x \oplus y) \oplus y) \oplus z.$$

Since

$(\neg(\neg x \oplus y) \oplus y) \oplus z = (\neg(\neg y \oplus x) \oplus x) \oplus z = \neg(\neg y \oplus x) \oplus (x \oplus z)$

by (M5) and (M4), we get

(4) $\neg(\neg x \oplus y) \oplus (y \oplus z) = \neg(\neg y \oplus x) \oplus (x \oplus z)$.

For $y = \neg 0$ in (M5) we compute

$$\neg(\neg x \oplus \neg 0) \oplus \neg 0 = \neg(\neg\neg 0 \oplus x) \oplus x.$$

From this, applying (M6), (M2) and (3), we have

$$\neg 0 = \neg(0 \oplus x) \oplus x,$$

which, using (3) again, give us

(5) $\neg 0 = \neg x \oplus x$.

Now, put $\neg y$ instead of y and y instead of z in (4) to obtain

$$\neg(\neg x \oplus \neg y) \oplus (\neg y \oplus y) = \neg(\neg\neg y \oplus x) \oplus (x \oplus y).$$

By (5) and (M2) we reduce this to

$$\neg(\neg x \oplus \neg y) \oplus \neg 0 = \neg(y \oplus x) \oplus (x \oplus y).$$

Using (M6), we have

(6) $\neg 0 = \neg(y \oplus x) \oplus (x \oplus y)$.

Finally, we are going to prove that $x \oplus y = y \oplus x$. Using (3) and (M2) we compute

$$x \oplus y = 0 \oplus (x \oplus y) = \neg\neg 0 \oplus (x \oplus y).$$

Further, with (6) and (M5),

$$\neg\neg 0 \oplus (x \oplus y) = \neg(\neg(y \oplus x) \oplus (x \oplus y)) \oplus (x \oplus y) =$$
$$= \neg(\neg(x \oplus y) \oplus (y \oplus x)) \oplus (y \oplus x)$$

which is equal to $\neg\neg 0 \oplus (y \oplus x)$ by (6), where x is substituted by y and vice versa. Since $\neg\neg 0 \oplus (y \oplus x) = 0 \oplus (y \oplus x) = y \oplus x$, by (M2) and (3), we are done. □

THEOREM

The axioms (M1), (M2), (M4), (M5) and (M6) are independent.

Proof

Denote by B the two-element set $\{0, 1\}$.

(I) Consider an algebra $(B; \oplus, \neg, 0)$ where $\oplus$ is a constant operation: $x \oplus y = 0$ for all $x, y \in B$ and $\neg 0 = 0$, $\neg 1 = 1$. One can easily check that this algebra satisfies (M2), (M4), (M5), (M6) but not (M1) since $1 \oplus 0 = 0 \neq 1$.

(II) Now, let $(B; \oplus)$ be a join-semilattice and $\neg x = 1$ for all $x \in B$. Then $(B; \oplus, \neg, 0)$ satisfies (M1), (M4), (M5) and (M6) but, trivially, not (M2).

(III) Let $\mathcal{C} = (\{0, 1, 2\}; \oplus, \neg, 0)$ be an algebra of type $(2, 1, 0)$ where the operation $\oplus$ and $\neg$ are defined by the following tables:

$\oplus$	0	1	2
0	0	1	1
1	1	1	2
2	2	1	2

x	0	1	2
$\neg x$	1	0	2

Evidently $\mathcal{C}$ satisfies (M1), (M2), (M6). We can show that (M4) is not satisfied: take $x = 0$, $y = 1$, $z = 2$. Then

$$(0 \oplus 1) \oplus 2 = 1 \oplus 2 = 2 \neq 1 = 0 \oplus 2 = 0 \oplus (1 \oplus 2).$$

It remains to prove that $\mathcal{C}$ satisfies (M5).

(a) If $y = 0$ then (M5) is reduced to $x = \neg(1 \oplus x) \oplus x$ which is plain to check.

(b) If $y = 1$ then (M5) is $1 = \neg(0 \oplus x) \oplus x$ which one can easily check.

(c) For $y = 2$ is (M5) as follows:

$\neg(\neg x \oplus 2) \oplus 2 = \neg(\neg 2 \oplus x) \oplus x$ which also holds for each $x \in \{0, 1, 2\}$.

(IV) Let $\mathcal{D} = (\{0, 1, 2\}; \oplus, \neg, 0)$ be an algebra of type $(2, 1, 0)$ where the operations $\oplus$ and $\neg$ are defined by the following tables:

$\oplus$	0	1	2
0	0	1	0
1	1	1	1
2	2	1	2

x	0	1	2
$\neg x$	1	0	2

Evidently $\mathcal{D}$ satisfies (M1), (M2) and (M6). We can show that (M5) is not satisfied: take $x = 0$, $y = 2$. Then

$$\neg(\neg 0 \oplus 2) \oplus 2 = \neg(1 \oplus 2) \oplus 2 = \neg 1 \oplus 2 = 0 \neq 2 = \neg\neg 2 = \neg(\neg 2 \oplus 0) \oplus 0.$$

It remains to prove that $\mathcal{D}$ satisfies (M4).

(a) If $z = 0$ or $z = 2$ then (M4) is reduced to $x \oplus y = x \oplus y$ which is always true.

(b) If $z = 1$ then (M4) is $(x \oplus y) \oplus 1 = x \oplus (y \oplus 1)$ which is evidently true.

(V) Finally, let $(B; \oplus)$ be a join-semilattice and $\neg$ be the identity mapping on B. Then clearly (M1), (M2) and (M4) are satisfied. To prove (M5) we mention that for $x = y$ it is trivial as well as for the general case $\{x, y\} = \{0, 1\}$ since

$$\neg(\neg 1 \oplus 0) \oplus 0 = 1 \oplus 0 = 1 = 1 \oplus 1 = \neg(\neg 0 \oplus 1) \oplus 1.$$

It remains to show that (M6) is violated. For this, take $x = 1$. Then clearly the left-hand side of (M6) equals to 1 but the right-hand side is 0. □

Open problem

It is an open problem whether another axiom, different from (M3), can be removed from (M1)–(M6) to obtain an axiomatic system of MV-algebras. After a preliminary inspection, the author conjectures that this is not possible.

REFERENCES

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