T. Kuhr, V. Vychodil: Representing Graded Attribute Implications by Fuzzy Logic Programs

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Introduction

This poster contributes to the field of reasoning with graded if-then rules and presents a link between two logic systems: fuzzy logic programming (FLP) in sense of Vojtáš and fuzzy attribute logic (FAL) in sense of Belohlavek and Vychodil. Both the systems play an important role in computer science and artificial intelligence as they can be used for approximate knowledge representation and inference, description of dependencies found in data, representing approximate constraints in relational similarity-based databases, etc.

Although these systems are technically different and were developed to serve different purposes, they share common features:

- they are based on residuated structures of truth degrees,
- use truth-functional interpretation of logical connectives,
- can describe if-then dependencies in problem domains while treating with inexact matches,
- models of theories form closure systems and semantic entailment can be expressed by means of least models.

It is therefore appealing to look closer at their mutual relationship. Furthermore, a possible link between the two systems can bring forth some new results.

Residuated Structures

We consider a complete lattice $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$ with L representing a set of degrees (bounded by 0 and 1 representing the full falsity and full truth) and the corresponding order $\leq$. In order to express truth functions of general logical connectives, we assume that $\mathbf{L}$ is equipped by a collection of pairs of the form $\langle \otimes, \rightarrow\rangle$ such that $\langle L, \otimes, 1\rangle$ is a commutative monoid, and $\otimes$ and $\rightarrow$ satisfy the adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{1}$$

for any $a, b, c \in L$. As usual, $\otimes$ and $\rightarrow$ serve as truth functions of binary logical connectives "fuzzy conjunction" and "fuzzy implication". If $\otimes$ and $\rightarrow$ satisfy (1), then $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ is called a complete residuated lattice. Note that there are complete lattices that cannot be equipped with adjoint operations. On the other hand, there are complete lattices with multiple possible adjoint operations.

An $\mathbf{L}$-set A in universe U is a map $A\colon U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". L^U denotes the collection of all $\mathbf{L}$-sets in U. By $\{{}^{a}/u\}$ we denote an $\mathbf{L}$-set A in U such that $A(u) = a$ and $A(v) = 0$ for $v \neq u$. An $\mathbf{L}$-set $A \in L^U$ is called crisp if $A(u) \in \{0, 1\}$ for all $u \in U$.

For $a \in L$ and $A \in \mathbf{L}^U$, we define $\mathbf{L}$-sets $a \otimes A$ by $(a \otimes A)(u) = a \otimes A(u)$ for all $u \in U$. For $A, B \in L^U$, we define a subsethood degree of A in B:

$$S(A, B) = \textstyle\bigwedge_{u \in U}\bigl(A(u) \rightarrow B(u)\bigr).$$

Fuzzy Attribute Logic

Let $\mathbf{L}$ be a complete residuated lattice and Y be a nonempty set of attributes. A fuzzy attribute implication (FAI) is an expression $A \Rightarrow B$, where $A, B \in L^Y$. For an $\mathbf{L}$-set $M \in L^Y$ of attributes, we define a degree $||A \Rightarrow B||_M \in L$ to which $A \Rightarrow B$ is true in M by

$$||A \Rightarrow B||_M = S(A, M)^* \rightarrow S(B, M),$$

where * is an unary operation on L satisfying: (i) $1^* = 1$, (ii) $a^* \leq a$, (iii) $(a \rightarrow b)^* \leq a^* \rightarrow b^*$, and (iv) $a^{**} = a^*$ for all $a, b \in L$. The operation * is called a hedge and can be seen as a truth function of a connective "very true". We use * as a parameter of the interpretation of $A \Rightarrow B$.

We consider semantic entailment based on satisfaction of FAIs in models. Recall that M is a model of an $\mathbf{L}$-set T of FAIs if $T(A \Rightarrow B) \leq ||A \Rightarrow B||_M$ for all $A, B \in L^Y$. Denoting the set of all models of T by $\mathrm{Mod}(T)$, we define a degree $||A \Rightarrow B||_T$ to which $A \Rightarrow B$ semantically follows from T by

$$||A \Rightarrow B||_T = \textstyle\bigwedge_{M \in \mathrm{Mod}(T)} ||A \Rightarrow B||_M.$$

Fuzzy Logic Programming

According to Vojtáš, we consider a complete lattice $\mathbf{L}$ on the unit interval with multiple adjoint operations. A language $\mathcal{L}$ for a fuzzy logic program (FLP) is given by a finite nonempty set R of relation symbols, a finite set F of function symbols and a denumerable set of variables. Moreover, $\mathcal{L}$ also contains symbols for binary logical connectives

- $\wedge_1, \wedge_2, \ldots$ (fuzzy conjunctions),
- $\Rightarrow_1, \Rightarrow_2, \ldots$ (fuzzy implications),
- and symbols for aggregations $\mathbb{ag}_1, \mathbb{ag}_2, \ldots$

Terms, formulas, substitutions and theories are defined as usual. A definite program P is a theory such that

- finitely many formulas have assigned a nonzero degree,
- assigned degrees are rational numbers from $[0, 1]$,
- each formula with a nonzero degree is either atomic or in the form $\psi \Leftarrow \varphi$, where ψ is atomic and φ does not contain any implication.

Notions of a Herbrand universe, a Herbrand base and a structure for P are defined as usual. For a structure M, $M(\chi)$ is interpreted as a degree to which the atomic ground formula χ is true under M. The notion of a formula being true in M can be extended to all formulas in an obvious way.

Structure M is called a model for theory T if $T(\chi) \leq M(\chi)$ for each formula χ. The collection of all models of T will be denoted by $\mathrm{Mod}(T)$. A pair $\langle a, \theta\rangle$ consisting of $a \in (0, 1]$ and a substitution θ is a correct answer for a program P and a query φ if $M(\varphi\theta) \geq a$ for each $M \in \mathrm{Mod}(P)$.

Representing FAIs by FLPs

Let $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ be a complete residuated lattice on the real unit interval in this section. We also consider only FAIs of the form $A \Rightarrow B$, where both A and B are finite (i.e., there are finitely many attributes $y \in Y$ such that $A(y) > 0$ or $B(y) > 0$). We call these fuzzy attribute implications finitely presented FAIs.

Theorem 1. *For each finite theory T of finitely presented FAIs and a finitely presented $A \Rightarrow B$ there is a definite program P such that $||A \Rightarrow B||_T \geq a$ iff for each attribute $y \in Y$ such that $a \otimes B(y) > 0$, the pair $\langle a \otimes B(y), \emptyset\rangle$ is a correct answer for the program P and atomic formula y.*

Proof. We can assume that T is crisp. If it is not, we can take a corresponding crisp theory T' given by

$$T' = \{A \Rightarrow T(A \Rightarrow B) \otimes B \,|\, T(A \Rightarrow B) \otimes B \nsubseteq A\}.$$

We consider a language $\mathcal{L}$ with only nullary relation symbols $y_1, y_2, \ldots, y_k$ that correspond to attributes which appear in the FAIs from T to a nonzero degree and a nullary relation symbol e. Clearly, R is a finite set and the Herbrand base $\mathcal{B}_P$ of any program P in $\mathcal{L}$ is equal to R. In addition, we assume that $\mathcal{L}$ contains the following logical connectives and aggregations:

- $\Rightarrow$ (interpreted by the residuum $\rightarrow$ in $\mathbf{L}$),
- $\wedge$ (interpreted by the infimum $\wedge$ in $\mathbf{L}$),
- a unary aggregation $\mathbb{ts}$ (interpreted by an idempotent truth-stressing hedge *),
- for each rational $a \in (0, 1]$ a binary aggregation $\mathbb{sh}_a$ (interpreted by $M^\sharp(\mathbb{sh}_a(\varphi, \psi)) = (a \rightarrow M^\sharp(\varphi)) \wedge M^\sharp(\psi)$).

Since all FAIs in T are finitely presented, for any $C \Rightarrow D \in T$ and arbitrary attribute $y \in Y$, we can consider a rule of FLP

$$y \Leftarrow \mathbb{ts}\bigl(\mathbb{sh}_{C(z_1)}(z_1, e) \wedge \cdots \wedge \mathbb{sh}_{C(z_n)}(z_n, e)\bigr), \tag{2}$$

where $z_1, \ldots, z_n$ are exactly the attributes which belong to C to a nonzero degree provided that $C \neq \emptyset$. In the special case of $C = \emptyset$, we can let (2) be just the fact y. Notice that (2) is a properly defined rule of a definite program written in a language $\mathcal{L}$. We denote the rule (2) by $y \Leftarrow C$.
Moreover, for any finite crisp T of finitely presented FAIs, we can consider an $\mathbf{L}$-set of rules P_T defined by

$$P_T(y \Leftarrow C) = \textstyle\bigvee\{D(y) \,|\, D \in L^Y \text{ such that } C \Rightarrow D \in T\}$$

for all $y \in Y$ and $C \in L^Y$. Furthermore, we put $P_T(e) = 1$. Clearly, P_T is a definite program in $\mathcal{L}$ in sense of Vojtáš.

The proof then continues by observing that $||A \Rightarrow B||_T = a > 0$ iff $||A \Rightarrow a \otimes B||_T = 1$ iff $||\emptyset \Rightarrow a \otimes B||_{T \cup \{\emptyset \Rightarrow A\}} = 1$ iff $a \otimes B(y) \leq ||\emptyset \Rightarrow \{{}^{1}/y\}||_{T \cup \{\emptyset \Rightarrow A\}}$ for all $y \in Y$ such that $B(y) > 0$. The latter is true iff for each $y \in Y$ such that $B(y) > 0$, the pair $\langle a \otimes B(y), \emptyset\rangle$ is a correct answer for the program $P_{T \cup \{\emptyset \Rightarrow A\}}$ and query y. □

Example

Let $\mathbf{L}$ be the standard Łukasiewicz structure and * be the identity. Consider a set of attributes of cars $Y = \{lA, lM, hAT, hFE, hP\}$ which mean: "a car has low age", "has low mileage", "has automatic transmission", "has high fuel economy" and "has high price" respectively. Let T being a set containing the following FAIs over Y:

$$\bigl\{{}^{0.7}/lA, {}^{0.9}/lM, {}^{0.4}/hAT\bigr\} \Rightarrow \bigl\{{}^{0.6}/hFE, {}^{0.9}/hP\bigr\},$$
$$\bigl\{{}^{0.8}/lA\bigr\} \Rightarrow \bigl\{{}^{0.7}/lM\bigr\}.$$

Using Theorem 1, we can find a FLP P_T that corresponds to FAIs from T. The program P_T will contain the following rules:

$$hFE \overset{0.6}{\Leftarrow} \mathbb{ts}\bigl(\mathbb{sh}_{0.7}(lA, e) \wedge \mathbb{sh}_{0.9}(lM, e) \wedge \mathbb{sh}_{0.4}(hAT, e)\bigr),$$
$$hP \overset{0.9}{\Leftarrow} \mathbb{ts}\bigl(\mathbb{sh}_{0.7}(lA, e) \wedge \mathbb{sh}_{0.9}(lM, e) \wedge \mathbb{sh}_{0.4}(hAT, e)\bigr),$$
$$lM \overset{0.7}{\Leftarrow} \mathbb{ts}\bigl(\mathbb{sh}_{0.8}(lA, e)\bigr).$$

Obviously, the aggregator $\mathbb{ts}$ can be omitted.

Now, we can use the results from FLP and Theorem 1 to characterize $||A \Rightarrow B||_T$ using computed answers for program $P_{T \cup \{\emptyset \Rightarrow A\}}$ and queries $y \in Y$ with $B(y) > 0$.

For example, a user asks a question "How much expensive are quite new cars with automatic transmission?", i.e., more precisely "To what degree $a \in L$, is the FAI $\{{}^{0.6}/lA, {}^{1}/hAT\} \Rightarrow \{{}^{1}/hP\}$ true in T?". To get the answer, we first extend P_T to $P_{T \cup \{\emptyset \Rightarrow A\}}$ by adding facts $lA \overset{0.6}{\Leftarrow}$ and $hAT \overset{1}{\Leftarrow}$ to the program. Then, we can easily compute an answer to query hP (all substitutions are $\emptyset$):

$$hP,$$
$$0.9 \otimes \bigl(\mathbb{sh}_{0.7}(lA, e) \wedge \mathbb{sh}_{0.9}(lM, e) \wedge \mathbb{sh}_{0.4}(hAT, e)\bigr),$$
$$0.9 \otimes \bigl(\mathbb{sh}_{0.7}(lA, e) \wedge \mathbb{sh}_{0.9}(0.7 \otimes \mathbb{sh}_{0.8}(lA, e), e) \wedge \mathbb{sh}_{0.4}(hAT, e)\bigr),$$
$$0.9 \otimes \bigl(\mathbb{sh}_{0.7}(0.6, 1) \wedge \mathbb{sh}_{0.9}(0.7 \otimes \mathbb{sh}_{0.8}(0.6, 1), 1) \wedge \mathbb{sh}_{0.4}(1, 1)\bigr),$$
$$0.9 \otimes \bigl(0.7 \rightarrow 0.6 \wedge 0.9 \rightarrow (0.7 \otimes (0.8 \rightarrow 0.6)) \wedge 0.4 \rightarrow 1\bigr),$$
$$0.5.$$

Using Theorem 1 and the computed answer $\langle 0.5, \emptyset\rangle$, we immediately get $||\{{}^{0.6}/lA, {}^{1}/hAT\} \Rightarrow \{{}^{1}/hP\}||_T = 0.5$.

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Conclusion

We have shown that fuzzy attribute implications can be reduced to fuzzy logic programs and semantic entailment of fuzzy attribute implications can be described via correct answers for fuzzy logic programs and queries. The results have shown a new theoretical insight and a link of two branches of rule-based reasoning methods.