On (Logical) Models for Reasoning under Uncertainty, Fuzziness and Truthlikeness

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Outline

- Introduction: uncertainty, fuzziness and truthlikeness

- Some logical approaches to reason under uncertainty

  $\Rightarrow$ A fuzzy logic approach

- Truthlikeness and similarity-based reasoning

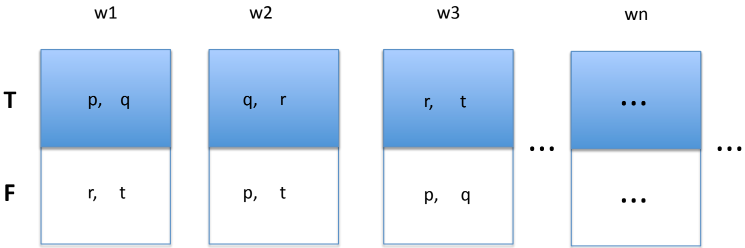
Uncertainty, fuzziness, truthlikeness

Possible worlds scenario: W

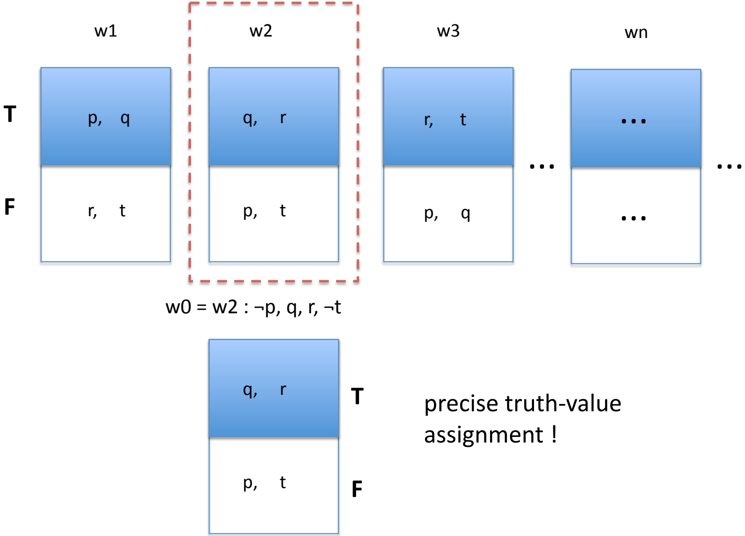
Ideal situation:
(i) complete information about which is the *real world* w_0
(ii) precise concepts: in any world, either $w \models \varphi$ or $w \models \neg\varphi$

$$Truth = \{\varphi \mid w_0 \models \varphi\} \quad Falsity = \{\psi \mid w_0 \models \neg\psi\}$$

precise concepts



precise concepts + complete information



precise truth-value assignment !

Uncertainty, fuzziness, truthlikeness

Some more realistic situations:

Uncertainty, fuzziness, truthlikeness

Some more realistic situations:

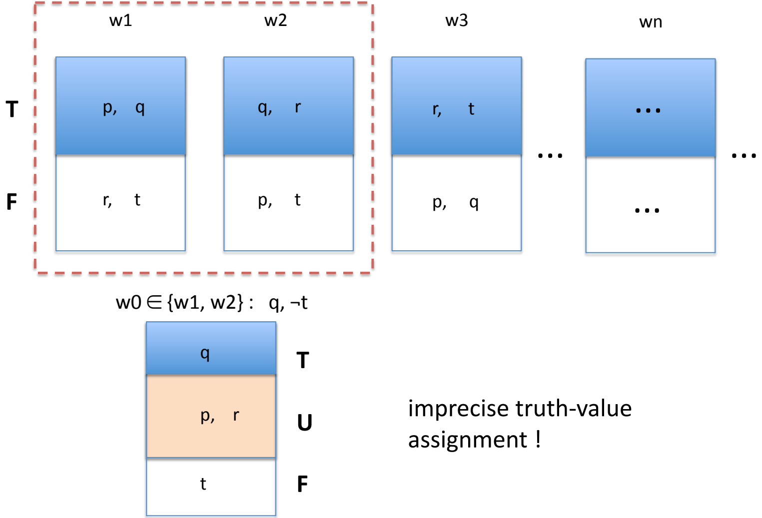
Uncertainty about w_0: incomplete information but still precise concepts

- the real world is in $K \subset W$

$$Truth = \{\varphi \mid \forall w \in K, w \models \varphi\} \quad Falsity = \{\psi \mid \forall w \in K, w \models \neg\psi\}$$

$$Undecided = \{\varphi \mid \varphi \notin Truth, \varphi \notin Falsity\}$$

precise concepts + incomplete information



imprecise truth-value assignment !

Uncertainty, fuzziness, truthlikeness

Uncertainty about w_0: a more informed scenario

- w_0 as a random variable with a probability function $\pi : W \to [0, 1]$
  how likely is that φ is true? $\mu(\varphi) = \sum_{w \models \varphi} \pi(w) \in [0, 1]$

Uncertainty, fuzziness, truthlikeness

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$$Undecided = \{\varphi \mid \mu(\varphi) = \alpha_1\} \cup \{\varphi \mid \mu(\varphi) = \alpha_2\} \cup \ldots \cup \{\varphi \mid \mu(\varphi) = \alpha_k\}$$

$$0 < \alpha_1 < \alpha_2 < \ldots \alpha_k < 1$$

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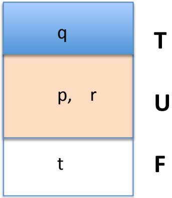
precise concepts + probabilistic information

$w0 \in \{w1, w2\} = K \subseteq W$

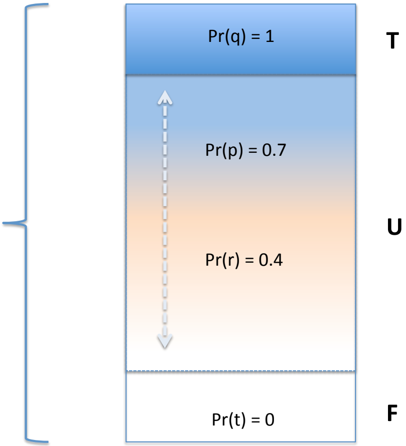


+

$Pr: K \longrightarrow [0, 1]$

$Pr(\phi_i) = \sum\{w: w(\phi_i) = 1\} = \alpha_i$

graded uncertainty



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- similar refined structures with other representation models (plausibility orderings, belief functions, . . .)
- logics of (numerical) belief: probabilistic, possibilistic, DS, etc.)
  - non truth-functional

Uncertainty vs. fuzziness

Fuzziness:
(i) complete information: the real world is w_0
(ii) gradual concepts: in any world, $w(\varphi) \in [0, 1]$

many-valued worlds, intermediate degrees of truth:

$$0 \leq truth(\varphi) = w_0(\varphi) \leq 1$$

Uncertainty vs. fuzziness

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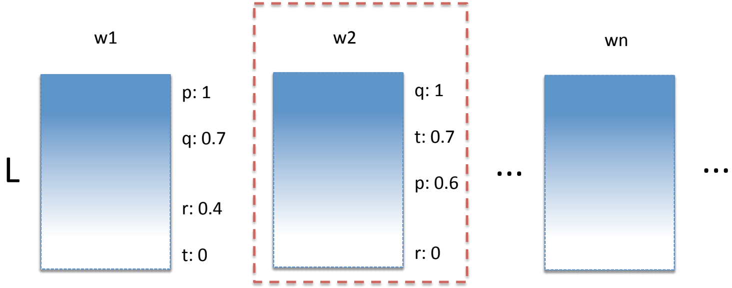
Mathematical fuzzy logics (after [Hájek, 1998]) :

- formal systems (syntax, semantics, complete axiomatizations, proof theory, etc...)
- $[0, 1]$: usual choice of truth-value set (standard semantics)
- truth-functionality assumption
- logics of comparative truth: $w(\varphi \to \psi) = 1$ iff $w(\varphi) \leq w(\psi)$

fuzzy concepts + complete information



w0 = w2

q = 1,
t = 0.7,
p = 0.6,
r = 0

precise truth-value assignment !

Uncertainty and Fuzziness

(Epistemic) uncertainty logics $\longleftrightarrow$ partial graded Belief

Fuzzy logics $\longleftrightarrow$ partial, graded Truth

partial belief $\neq$ partial truth

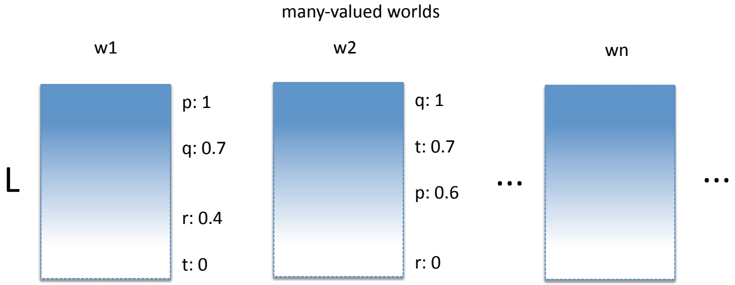
Logics of Belief	Logics of Fuzziness
Boolean truth values	Intermediate truth values
Degrees of belief	Degrees of truth
Induced by lack of information	Unavoidable graduality in concepts
Not fully compositional	Fully compositional (might)

Logic, Uncertainty and Fuzziness

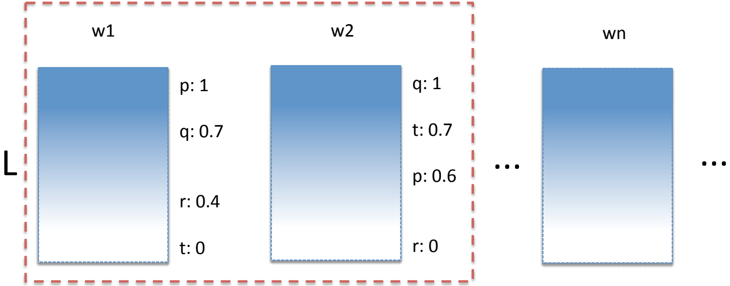
Even more complex scenarios : incomplete information + gradual concepts

$\Rightarrow$ uncertainty on the (many-valued) truth status of propositions

fuzzy concepts + incomplete information



fuzzy concepts + incomplete information



$w0 \in \{w1, w2\}$:

$0.6 \leq p \leq 1,$
$0.7 \leq q \leq 1,$
$0 \leq r \leq 0.4,$
$0 \leq t \leq 0.7$

imprecise truth-value assignment !

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Even more complex scenarios : incomplete information + gradual concepts

$\Rightarrow$ uncertainty on the (many-valued) truth status of propositions

$\Rightarrow$ uncertainty measures on (many-valued) possible worlds

e.g. given $p : W \to [0, 1]$ probability distribution, define

$$\mu(\varphi) = \sum_{w \in W} p(w) \cdot w(\varphi)$$

(average or expected truth-value of φ in W)

$\Rightarrow$ logics to reason about the uncertainty of fuzzy events (generalized probabiliy, necessity, belief functions, etc.)

Truthlikeness

Truthlikeness $\neq$ Uncertainty, Fuzziness

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φ_1: there are 150 steps to the top of *The Tower of Olomouc Town Hall*

φ_2: there are 300 steps to the top of *The Tower of Olomouc Town Hall*

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In the real world w_0 both are false (there are 152!),

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... but clearly φ_1 provides a more accurate description of w_0 than φ_2.

Indeed, 150 is more similar to 152 than 300.

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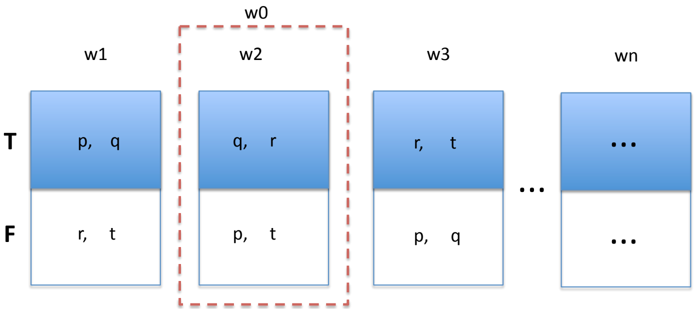
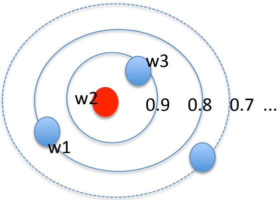
"φ_1 is closer to be true (more truth-like) than φ_2"

precise concepts + similarity relation



+

$$S : W \times W \longrightarrow [0, 1]$$



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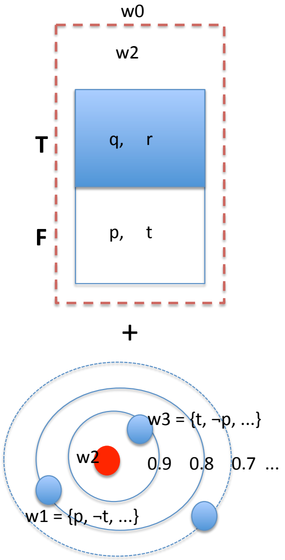

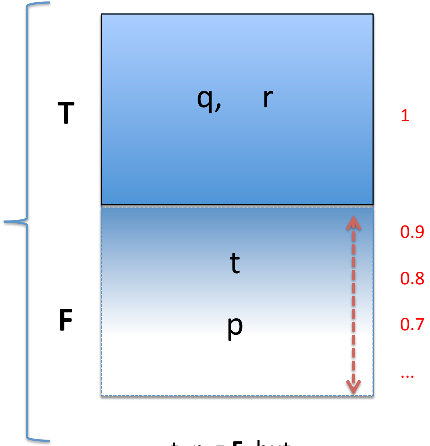

t, p $\in$ **F**, but
t is more truth-like than p !

Truthlikeness

- (G. Oddie, Stanford Encyclopedia of Philosophy)

  Truthlikeness: *"... classify propositions according to their closeness to the truth, their degree of truthlikeness or verisimilitude ... give an adequate account of the concept and to explore its logical properties and its applications . . . to epistemology and methodology"*

  - Popper, Tichý, Hilpinen, Niiniluoto, ...

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- A further (independent) dimension to be additionally considered to models dealing with imperfect information (uncertainty, fuzziness, nonmonotonicity, ect.)

Outline

- Introduction: uncertainty, fuzziness and truthlikeness
- Some logical approaches to reason under uncertainty
  - Measures of uncertainty: a brief overview
  - Probabilistic and possibilistic logics
  - A fuzzy modal approach
- Truthlikeness and similarity-based reasoning

Graded representation of uncertainty

B: set of events (Boolean algebra)

logical setting: $\mathbf{B} = \mathcal{L}/\equiv$
events = propositions (mod. logical equivalence)
$\top$ always true event,
$\bot$ always false event

Uncertainty, belief measures $g : \mathcal{L} \to [0, 1]$

(1) $g(\top) = 1, g(\bot) = 0$
(2) $g(\varphi) \leq g(\psi)$, if $\vdash \varphi \to \psi$

Fuzzy measures (Sugeno) or Plausibility measures (Halpern)

$g(\varphi)$ quantifies an agent's confidence/belief on φ being true

Uncertainty measures: a typology

$$g : \mathcal{L} \to [0, 1]$$

(1) $g(\top) = 1, g(\bot) = 0$
(2) $g(\varphi) \leq g(\psi)$, if $\vdash \varphi \to \psi$

General properties

$g(\varphi \wedge \psi) \leq \min(g(\varphi), g(\psi))$
$g(\varphi \vee \psi) \geq \max(g(\varphi), g(\psi))$

They are NOT compositional !

Uncertainty measures: a typology

(Finitely additive) Probability measures

(3) finite additivity: $P(\varphi \vee \psi) = P(\varphi) + P(\psi)$, whenever $\vdash \varphi \wedge \psi \equiv \bot$

- $P(\neg\varphi) = 1 - P(\varphi)$ (auto-dual)
- $P(\varphi_1 \vee \ldots \vee \varphi_n) = \sum_{i=1,n} P(\varphi_i) - \sum_{i<j} P(\varphi_i \wedge \varphi_j) + \ldots + (-1)^{n-1} P(\varphi_1 \wedge \ldots \wedge \varphi_n)$

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Possibility and Necessity measures

(3') Possibility: $\Pi(\varphi \vee \psi) = \max(\Pi(\varphi), \Pi(\psi))$

(3") Necessity: $N(\varphi \wedge \psi) = \min(N(\varphi), N(\psi))$

Dual pairs of measures (N, Π): when $\Pi(\varphi) = 1 - N(\neg\varphi)$

$\pi : \Omega \to [0,1]$ s.t. $N(\varphi) = \inf_{\omega(\varphi)=0} 1 - \pi(\omega)$, $\Pi(\varphi) = \sup_{\omega(\varphi)=1} \pi(\omega)$

Uncertainty measures: a typology

Dempster-Shafer Belief and Plausibility functions

(3') Belief function: for each n,

$$bel(\varphi_1 \vee \ldots \vee \varphi_n) \geq \sum_{i=1,n} bel(\varphi_i) - \sum_{i<j} bel(\varphi_i \wedge \varphi_j) + \ldots + (-1)^{n-1} bel(\varphi_1 \wedge \ldots \wedge \varphi_n)$$

in particular: $bel(\varphi \vee \psi) \geq bel\varphi + bel(\psi) - bel(\varphi \wedge \psi)$ (super-additivity)

(3") Plausibility function: for each n,

$$pl(\varphi_1 \vee \ldots \vee \varphi_n) \leq \sum_{i=1,n} pl(\varphi_i) - \sum_{i<j} pl(\varphi_i \wedge \varphi_j) + \ldots + (-1)^{n-1} pl(\varphi_1 \wedge \ldots \wedge \varphi_n)$$

in particular: $pl(\varphi \vee \psi) \leq pl\varphi + pl(\psi) - pl(\varphi \wedge \psi)$ (sub-additivity)

Uncertainty measures: a typology

Lower and Upper probabilities

Let $\mathcal{P} = \{P_i\}_{i \in I}$ a family of probability measures over the same space $\mathcal{L}$

$\mathcal{P}^*(\varphi) = \sup\{P_i(\varphi) \mid i \in I\}$ – upper probability
$\mathcal{P}_*(\varphi) = \inf\{P_i(\varphi) \mid i \in I\}$ – lower probability

$\mu^* : \mathcal{L} \to [0,1]$ is an upper probability iff it is a measure satisfying that for all natural numbers m, n, k, and all $\varphi_1, \ldots, \varphi_m$, if $\{\{\varphi_1, \ldots, \varphi_m\}\}$ is an (n,k)-cover of $(\varphi, \top)$, then

(3') $\quad k + n\mu^*(\varphi) \leq \sum_{i=1}^{m} \mu^*(\varphi_i)$.

μ_* is a lower probability iff ... analogously, replacing (3') by

(3") $\; k + n\mu_*(\varphi) \geq \sum_{i=1}^{m} \mu_*(\varphi_i)$.

Some logical approaches to reason about uncertainty

Brief overview of the basic features of different approaches in the literature (with many simplifications!)

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Possibilisitic logic (after Dubois-Prade et al.)

- Language: weighted formulas of the type

$$(\varphi, \alpha)$$

  where φ is a CPC formula, $\alpha \in [0, 1]$
- Semantics: given by possibility distributions $\pi : W \to [0, 1]$ on the set of interpretations

$$\pi \models (\varphi, \alpha) \text{ iff } N_\pi(\varphi) = \inf_{w(\varphi)=0} 1 - \pi(w) \ \geq \ \alpha$$

- Sound and complete axiomatizations and proof systems
- Many variants proposed; Applications to non-monotonic reasoning, theory change, belief merging, etc. ; Graphical models

Possibilisitic logic

φ is certain	$(\varphi, 1)$
φ is α-certain	(φ, α)
φ is unknown	$(\varphi, 0)$
φ is β-false	$(\neg\varphi, \beta)$
φ is false	$(\neg\varphi, 1)$

Automated deduction

- The proof method in PL, denoted $\vdash^r_{PL}$, is defined by refutation through resolution.

- Resolution rule:

$$\frac{(\neg p \vee q, \alpha), (p \vee r, \beta)}{(q \vee r, \min(\alpha, \beta))}$$

- $\Gamma \vdash^r_{PL} (\varphi, \alpha)$ iff we obtain a proof of $(\bot, \alpha)$ by successively applying the resolution rule in $\Gamma \cup (\neg\varphi, 1)$
  – Γ *and* $(\neg\varphi, 1)$ *put in clausal form* –

- Soundness and completeness [Dubois-Lang-Prade, 94]

$$\Gamma \models_{PL} (\varphi, \alpha) \text{ iff } \Gamma \vdash^r_{PL} (\varphi, \alpha)$$

Some logical approaches to reason about uncertainty

Halpern (et al.)'s approach

Defined on top of a system to reasoning about linear inequalities

- Language: built from CPC and likelihood formulas
  If $\varphi_1, ...\varphi_k$ are CPC formulas and $a_1, ..., a_k, b \in \mathbb{R}$ then

$$\Phi \ := \ a_1\ell(\varphi_1) + ... + a_k\ell(\varphi_k) \geq b$$

  is a basic likelihood formula. Extension when the φ_i's are also likelihood formulas, and close by $\wedge$ and $\neg$

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- (Probabilistic) Semantics given by the class of (probabilistic) Kripe models $(W, \pi, \{\mu_w\}_{w \in W})$, $\mu_w : \mathcal{U} \subseteq 2^W \to [0, 1]$

  $$M, w \models \Phi \text{ iff } a_1 \mu_w([\phi_1]) + \ldots + a_k \mu_w([\phi_k]) \geq b$$

- Complete axiomatizations for quite a lot of different classes of measure-based Kripke models (probabilities, possibilities, ranking functions, belief functions, upper and lower probabilities)

Some logical approaches to reason about uncertainty

Markovic, Ognjanovic et al. approach

- Defined on a two level language in a more standard modal logic way

- Language:
  For^C: $\varphi_1, ...\varphi_k$ CPC formulas
  For^P: basic P-formulas $P_{=a}\varphi$, $P_{\geq a}\varphi$; general P-formulas are Boolean combinations of basic P-formulas

$$P_{\geq a}\varphi \wedge \neg P_{\geq b}\psi \rightarrow P_{\geq c}\chi$$

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$$M \models P_{\geq a}\varphi \text{ iff } \mu_w([\phi]) \geq a$$

- Complete axiomatizations for many variants wrt different classes of Kripke models (cond. probabilities, possibilities, decomposable measures, etc.)

A fuzzy (modal) approach to reason about uncertainty

After (Hájek-G-Esteva, 95; Hájek, 98):

- introduce a modality P, s.t. for each classical proposition φ,

  $\mathbf{P}\varphi$ reads e.g. "φ **is probable**"

- $P\varphi$ is a gradual, fuzzy proposition: the higher is the probability of φ, the truer is $P\varphi$

- intuitive semantics: for φ a two-valued, crisp proposition one can define e.g.

  $$truth(P\varphi) = probability(\varphi)$$

  (which is different from $truth(\varphi) = probability(\varphi)$!!!)

A fuzzy (modal) approach to reason about uncertainty

Crucial observation: laws and computations with probability (and many other measures) can be expressed by well-known fuzzy logic truth-functions on $[0, 1]$.

$$\begin{aligned} Prob(A \cup B) &= Prob(A) + Prob(B) - Prob(A \cap B) \\ &= Prob(A) \oplus (Prob(B) \ominus Prob(A \cap B)) \end{aligned}$$

$$Prob(A \cap B) = Prob(A) \cdot Prob(A \mid B)$$

$$Nec(A \cap B) = \min(Nec(A), Nec(B))$$

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$$

$$
Nec(A \cap B) = \min(Nec(A), Nec(B))
$$

Idea: axioms of different uncertainty measures on φ's to be encoded as *axioms of suitable fuzzy logic* theories over the $P\varphi$'s

$$\left(\quad \cdots \right.$$

T-norm based fuzzy logics (Hájek, 1998)

Each (left-)continous t-norm $*$ defines a (prop.) calculus $PC(*)$:

Language:

- primary connectives: $\&$, $(\wedge,)$ $\rightarrow$, $\overline{0}$
- definable connectives: $\neg$, $\vee$, $(\wedge,)$ $\leftrightarrow$

[0, 1]-based Semantics: $e : Var \rightarrow [0, 1]$

- $e(\varphi \wedge \psi) = \min(e(\varphi), e(\psi))$
- $e(\varphi \& \psi) = e(\varphi) * e(\psi)$,
- $e(\varphi \rightarrow \psi) = e(\varphi) \Rightarrow e(\psi)$, where $\Rightarrow$ is the residuum of $*$

$$x * y \leq z \text{ iff } x \leq (y \Rightarrow z)$$

$([0, 1], *, \Rightarrow, \min, \max, 0, 1)$ residuated lattice

Hájek's BL logic

Axioms of BL:

A1	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
A2	$(\varphi \& \psi) \to \varphi$
A3	$(\varphi \& \psi) \to (\psi \& \varphi)$
A4	$(\varphi \& (\varphi \to \psi)) \to (\psi \& (\psi \to \varphi))$
A5	$(\varphi \to (\psi \to \chi)) \equiv ((\varphi \& \psi) \to \chi))$
A6	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
A7	$\overline{0} \to \varphi$

Inference Rule: modus ponens

- $\varphi \wedge \psi := \varphi \& (\varphi \to \psi)$ $\neg\varphi := \varphi \to \overline{0}$ $\overline{1} := \neg\overline{0}$

- BL-algebras $\langle L, *, \Rightarrow, \leq, 0, 1 \rangle$: bounded, pre-linear and divisible residuated lattices; if $L = [0, 1]$ then $*$ is a continuous t-norm

- Standard completeness: BL proves φ iff φ is a 1-tautology for every BL-algebra in [0, 1] (Hájek, 1998) (CEGT, 2000)

Main systems of fuzzy logic

Three main extensions of BL, corresponding to the three outstanding t-norms:

Łukasiewicz logic: Ł $=$ BL $+$ $\neg\neg\varphi \equiv \varphi$

- $e(\varphi \&_Ł \psi) = \max(0, e(\varphi) + e(\psi) - 1)$
  $e(\varphi \to_Ł \psi) = \min(1, 1 - e(\varphi) + e(\psi))$

Gödel logic: G $=$ BL $+$ $\varphi\&\varphi \equiv \varphi$

- $e(\varphi \&_G \psi) = \min(e(\varphi), e(\psi))$
  $e(\varphi \to_G \psi) = 1$ if $e(\varphi) \leq e(\psi)$, $= e(\psi)$ otherwise

Product logic: $\Pi =$ BL $+$ $(\Pi1), (\Pi2)$

- $e(\varphi \&_\Pi \psi) = e(\varphi) \cdot e(\psi)$
  $e(\varphi \to_\Pi \psi) = \min(1, e(\psi)/e(\varphi))$

Complete axiomatizations of 1-tautologies (cf. [Hájek, 98])

Definable connectives and truth functions in Łukasiewicz logic

Connective	Definition	Truth function
$\neg_L \varphi$	$\varphi \rightarrow_L \overline{0}$	$1 - x$
$\varphi \oplus \psi$	$\neg_L \varphi \rightarrow_L \psi$	$\min(1, x + y)$
$\varphi \ominus \psi$	$\varphi \& \neg_L \psi$	$\max(0, x - y)$
$\varphi \equiv_L \psi$	$(\varphi \rightarrow_L \psi) \& (\psi \rightarrow_L \varphi)$	$1 - \lvert x - y \rvert$
$\varphi \wedge \psi$	$\varphi \& (\varphi \rightarrow_L \psi)$	$\min(x, y)$
$\varphi \vee \psi$	$(\varphi \rightarrow_L \psi) \rightarrow_L \psi$	$\max(x, y)$

Expansions with truth-constants

After (Pavelka, 79):

Due to the residuation law, $e(\varphi) \leq e(\psi)$ iff $e(\varphi \to \psi) = 1$, T-norm based logics primarily deal with a notion of comparative truth:

"$T \models_{\mathrm{L}} \varphi \to \psi$": in the context of T, ψ is at least as true as φ

• how to capture the many-valuedness in reasoning with partial degrees of truth?

$\Rightarrow$ Logics expanded with truth-constants $\overline{r}$ for (some) $r \in [0, 1]$:

$$e(\overline{r} \to \varphi) = 1 \text{ iff } e(\varphi) \geq r$$

$$\cdots \Bigg)$$

$FP(CPC, Ł)$: a simple probability logic (HEG, 95), (Hájek, 98)

A two-level language:

(i) **Non-modal formulas:** φ, ψ, etc. , built from a set V of propositional variables $\{p_1, p_2, \dots p_n, \dots\}$ using the classical binary connectives $\wedge$ and $\neg$. The set of non-modal formulas will be denoted by $\mathcal{L}$.

(ii) **Modal formulas:** Φ, Ψ, etc. are built:
- from elementary modal formulas $P\varphi$, with $\varphi \in \mathcal{L}$
- using Lukasiewicz logic Ł connectives: $(\&_Ł, \rightarrow_Ł)$ and rational truth constants $\overline{r}$

Examples of formulas: $\overline{0.8} \rightarrow_Ł P(\varphi \wedge \chi)$, $P(\neg\varphi) \rightarrow_Ł P(\chi)$,

Non wff formulas: $\varphi \rightarrow_Ł P\psi$, $P(P\varphi \wedge P\chi)$

$FP(CPC, Ł)$: a two-level framework

events CPC

$$\neg(\psi \wedge \chi), \quad \varphi \wedge \psi \to \chi, \quad \varphi \vee (\psi \to \chi), \ \ldots$$

$FP(CPC, Ł)$: a two-level framework

Probabilistic atoms $\quad P\varphi, \quad P(\varphi \wedge \psi \to \chi), \quad P\neg(\psi \wedge \chi), \ldots$

events $\quad$ CPC

$$\neg(\psi \wedge \chi), \quad \varphi \wedge \psi \to \chi, \quad \varphi \vee (\psi \to \chi), \ldots$$

$FP(CPC, Ł)$: a two-level framework

$$P\varphi \equiv \overline{0.3}, \quad P(\varphi \wedge \psi) \to_{Ł} P\chi\ , \quad \overline{0.6} \to_{Ł} P(\psi \vee \varphi), \ \ldots$$

uncertainty — Łukasiewicz

Probabilistic atoms: $P\varphi, \quad P(\varphi \wedge \psi \to \chi)\ , \quad P\neg(\psi \wedge \chi), \ \ldots$

---

events — CPC

$$\neg(\psi \wedge \chi), \quad \varphi \wedge \psi \to \chi, \quad \varphi \vee (\psi \to \chi), \ \ldots$$

$FP(CPC, Ł)$: axiomatization

- The set of CPC tautologies
- Axioms of Łukasiewicz logic for modal formulas
- Probabilistic axioms:

(FP1)	$P(\varphi \to \psi) \to_Ł (P\varphi \to_Ł P\psi)$
(FP2)	$P(\varphi \vee \psi) \equiv (P\varphi \to_Ł P(\varphi \wedge \psi)) \to_Ł P\psi$
or equiv.	$P(\varphi \vee \psi) \equiv P\varphi \oplus (P\psi \ominus P(\varphi \wedge \psi))$
(FP3)	$P(\neg\varphi \mid \chi) \equiv \neg_Ł P(\varphi \mid \chi)$

- Deduction rules of FP(CPC, Ł) are *modus ponens* for $\to_Ł$ and
  (-) *necessitation* for P: from φ derive $P\varphi$

$FP(CPC, Ł)$: Semantics

Semantics: (weak) Probabilistic Kripke models $M = (W, e, \mu)$

- $e : W \times Var \to \{0, 1\}$
- $\mu : \mathcal{U} \subseteq 2^W \to [0, 1]$ probability such that the sets $[\varphi] = \{w \in W \mid \|\varphi\|_{M,w}) = 1\}$ are μ-measurable
- atomic modal formulas: $\|P\varphi\|_{M,w} = \mu([\varphi])$
- compound modal formulas: $\|\Phi\|_{M,w}$ is computed from atomic using Łukasiewicz connectives

$M = (W, e, \mu)$ is a model of Φ if for any $w \in W$, $\|\Phi\|_{M,w} = 1$

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$M = (W, e, \mu)$ is a model of Φ if for any $w \in W$, $\|\Phi\|_{M,w} = 1$

Alternatively: (strong) Probabilistic Kripke models $M = (W, e, \sigma)$ where

- $\sigma : W \to [0, 1]$ probability distribution $\Sigma_w \sigma(w) = 1$
- $\|P\varphi\|_{M,w} = \Sigma_w\{\sigma(w) \mid \|\varphi\|_{M,w}) = 1\}$

$FP(CPC, Ł)$: completeness

FS Completeness of $FP(CPC, Ł)$:
Let T a finite modal theory, Φ a modal formula. Then $T \vdash_{FP} \Phi$ iff any probabilistic model (W, e, μ) which is a model of T, is a model of Φ as well.

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A simplifying and clarifying reading:

$FP(CPC, Ł)$: completeness

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A simplifying and clarifying reading:

Weak (or strong) probabilistic models $M = (W, e,)$ are in 1-to-1 relation with probabilities on formulas $\mu : \mathcal{L} \to [0, 1]$ by $\mu(\varphi) = \|P\varphi\|_M$

(i) $\mu(\top) = 1$, $\mu(\bot) = 0$
(ii) $\mu(\varphi \vee \psi) = \mu(\varphi) + \mu(\psi) - \mu(\varphi \wedge \psi)$
(iii) $\mu(\varphi) = \mu(\psi)$ whenever φ and ψ are logically equivalent.

• Completeness: $T \vdash_{FP} \Phi$ iff any probability μ which satisfies the probabilistic expressions in T, also satisfies the probabilistic expression given by Φ.

An extension of $FP(CPC, Ł)$ to reason about conditional probabilities

Two issues to consider:

- to choose a suitable notion of conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \text{ if } P(B) > 0, \text{ but what if } P(B) = 0??$$

different solutions: non-standard probability, coherent conditional probability (de Finetti, Popper, Coletti-Scozzafava) :

$$P(A \cap B \mid C) = P(A \mid B, C) \cdot P(B \mid C)$$

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$$P(A \cap B \mid C) = P(A \mid B, C) \cdot P(B \mid C)$$

- to enlarge the arithmetical "machinery"
  $\Rightarrow$ expand Łukasiewicz logic with Product logic connectives

$$FP(CPC, Ł) \Rightarrow FCP(CPC, Ł\Pi\tfrac{1}{2})$$

$$\left(\quad \cdots \right.$$

The logic $Ł\Pi\frac{1}{2}$

- The logic $Ł\Pi$ ($= Ł + \Pi$) combines in a single framework
  (i) addition-related connectives & and $\rightarrow_L$ of Lukasiewicz logic Ł
  (ii) product-related connectives $\odot$ and $\rightarrow_\Pi$ of Product logic Π
- $Ł\Pi\frac{1}{2} = Ł\Pi$ + one truth-constant $\overline{\frac{1}{2}}$

Formulas:

- built from $(\&, \rightarrow_L) + (\odot, \rightarrow_\Pi)$ (+ truth-constant $\overline{\frac{1}{2}}$)
- many definable connectives: $\neg_L$, $\neg_\Pi$, $\wedge$, $\vee$, ...
- all rational truth-constants are also definable (in $Ł\Pi\frac{1}{2}$)

Examples:

$\varphi \odot \chi \rightarrow_L \psi$	truth(φ) $\cdot$ truth(χ) $\leq$ truth(ψ)
$\overline{0.7} \rightarrow_\Pi \psi$	$0.7 \leq$ truth(ψ)
$\neg_\Pi \neg_\Pi \varphi$	truth(φ) > 0

The logic $Ł\Pi\frac{1}{2}$ (2)

Axiom schemes and Rules for $Ł\Pi\frac{1}{2}$ (EGM, 2000), (Cintula, 2001)

- Axioms of Lukasiewicz logic for $(\&, \rightarrow_L)$ and of Product logic for $(\odot, \rightarrow_\Pi)$
- few additional axioms:

$$\varphi \odot (\psi \ominus \chi) \equiv (\varphi \odot \psi) \ominus (\varphi \odot \chi)$$
$$\Delta(\varphi \rightarrow_\Pi \psi) \equiv_L \Delta(\varphi \rightarrow_L \psi)$$
$$\neg_\Pi \varphi \rightarrow_L \neg_L \varphi$$

- modus ponens for $\rightarrow_L$
- necessitation for Δ: "from φ infer $\Delta\varphi$"

Finite strong completeness: for any finite theory T

$$T \vdash_{L\Pi\frac{1}{2}} \varphi \quad \text{iff} \quad e(\varphi) = 1 \text{ for each } e \text{ model of } T.$$

$$\cdots \right)$$

$FCP(CPC, Ł\Pi\frac{1}{2})$: a conditional probability logic (GM, 06)

P binary modality: $P(\varphi \mid \psi)$ reads "$\varphi \mid \psi$" is probable, where $\nvdash_{CPC} \neg\psi$

Axiomatization:

- The set of all $Taut(CPC)$
- Axioms of $Ł\Pi\frac{1}{2}$ for modal formulas
- Probabilistic axioms:

(FCP1)	$P(\varphi \to \psi \mid \chi) \to_L (P(\varphi \mid \chi) \to_L P(\psi \mid \chi))$
(FCP2)	$P(\varphi \vee \psi \mid \chi) \equiv ((P(\varphi \mid \chi) \to_L P(\varphi \wedge \psi \mid \chi)) \to_L P(\psi \mid \chi)$
(FCP3)	$P(\neg\varphi \mid \chi) \equiv \neg_L P(\varphi \mid \chi)$
(FCP4)	$P(\chi \mid \chi)$
(FCP5)	$P(\varphi \wedge \psi \mid \chi) \equiv P(\psi \mid \varphi \wedge \chi) \odot P(\varphi \mid \chi)$

- Deduction rules of FCP(CPC, $Ł\Pi\frac{1}{2}$) are those of $L\Pi\frac{1}{2}$ plus:
  - (-) *necessitation* for P: from φ derive $P(\varphi \mid \chi)$
  - (-) *substitution of equivalents*: from $\vdash_{CPC} \chi \equiv \chi'$, derive $P(\varphi \mid \chi) \equiv P(\varphi \mid \chi')$

Expressive power of FCP(CPC, ŁΠ)

Comparative statements

$\varphi\|\chi$ is more probable than $\psi\|\delta$	$P(\psi \mid \delta) \rightarrow_{Ł} P(\varphi \mid \chi)$

Numerical statements

probability of $\varphi\|\chi$ is at least 0.6	$\overline{0.6} \rightarrow_{Ł} P(\varphi \mid \chi)$
probability of $\varphi\|\chi$ is 0.5	$\overline{0.5} \equiv P(\varphi \mid \chi)$
$\varphi\|\chi$ has positive probability	$\neg_{\Pi}\neg_{\Pi} P(\varphi \mid \chi)$

Probabilistic independence statements

φ and δ are independent given χ	$P(\varphi \mid \chi \wedge \delta) \equiv_{Ł} P(\varphi \mid \chi)$

$FCP(CPC, \text{Ł}\Pi\frac{1}{2})$: semantics

Given by conditional probabilistic Kripke structures $\mathbf{M} = (\mathbf{W}, \mathcal{U}, \mathbf{e}, \mu)$:

- W arbitrary set of worlds, $e : W \times Atom \to \{0, 1\}$;
- $\mathcal{U} \subseteq 2^W$ Boolean algebra: $[\varphi] = \{w \in W \mid e(\varphi, w) = 1\} \in \mathcal{U}$
- $\mu : \mathcal{U} \times \mathcal{U}^0 \to [0, 1]$ coherent conditional probability
- $e(P(\varphi \mid \chi), w) = \mu([\varphi] \mid [\chi])$, if $[\chi] \neq \emptyset$
  $e(P(\varphi \mid \chi), w) =$ undefined, otherwise
- e is extended to compound modal formulas by $\text{Ł}\Pi\frac{1}{2}$ connectives

$\mathbf{M}$ is **safe** for a formula Φ if $e(\Phi, w)$ is defined (for all w)

Finite strong completeness wrt safe models

Examples

- $\{\varphi \rightarrow \psi, \overline{0.6} \rightarrow_{Ł} P(\varphi \mid \chi)\} \vdash_{FCP} \overline{0.6} \rightarrow_{Ł} P(\psi \mid \chi)$

$A \subseteq B$ and $\mu(A \mid C) \geq 0.6$ implies $\mu(B \mid C) \geq 0.6$

Examples

- $\{\varphi \to \psi, \overline{0.6} \to_{Ł} P(\varphi \mid \chi)\} \vdash_{FCP} \overline{0.6} \to_{Ł} P(\psi \mid \chi)$

$A \subseteq B$ and $\mu(A \mid C) \geq 0.6$ implies $\mu(B \mid C) \geq 0.6$

(i) $\varphi \to \psi \vdash_{FCP} P(\varphi \to \psi \mid \chi)$

(ii) $P(\varphi \to \psi \mid \chi) \vdash_{FCP} P(\varphi \mid \chi) \to_{Ł} P(\psi \mid \chi)$

(iii) $P(\varphi \mid \chi) \to_{Ł} P(\psi \mid \chi), \overline{0.6} \to_{Ł} P(\varphi \mid \chi) \vdash_{FCP} \overline{0.6} \to_{Ł} P(\psi \mid \chi)$

Examples

- $\{\varphi \to \psi, \overline{0.6} \to_Ł P(\varphi \mid \chi)\} \vdash_{FCP} \overline{0.6} \to_Ł P(\psi \mid \chi)$

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- $\{\neg(\varphi \wedge \psi), \overline{0.6} \to_Ł P(\varphi \mid \chi), \overline{0.3} \to_Ł P(\psi \mid \chi)\} \vdash_{FCP} \overline{0.9} \to_Ł P(\varphi \vee \psi \mid \chi)$

$A \cap B = \emptyset$, $\mu(A) \geq 0.6$, $\mu(B) \geq 0.3$ implies $\mu(A \cup B \mid C) \geq 0.9$

Examples

- $\{\varphi \to \psi, \overline{0.6} \to_Ł P(\varphi \mid \chi)\} \vdash_{FCP} \overline{0.6} \to_Ł P(\psi \mid \chi)$

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Probability computations = logical deductions !

Further logics for other uncertainty models

- Necessity and/or possibility logics $FN(CPC, Ł)$

$N\varphi$: φ is certain, necessary

...

(FN2) $\quad N(\varphi \wedge \psi) \equiv N\varphi \wedge_L N\psi$

...

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$N\varphi$: φ is certain, necessary

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(FN2) $\quad N(\varphi \wedge \psi) \equiv N\varphi \wedge_L N\psi$
. . .

- DS belief functions logic $FB(CPC, Ł\Pi\frac{1}{2}) = FP(S5, Ł\Pi\frac{1}{2})$

$B\varphi$: φ is believed as $P(\Box\varphi)$: $\Box\varphi$ is probable

- Uncertainty logics for fuzzy events:

  - generalized probability measures (states)
  - generalized necessity measures
  - generalized belief fucntions

Outline

- Introduction: uncertainty, fuzziness and truthlikeness
- Some logical approaches to reason under uncertainty
- Truthlikeness and similarity-based reasoning
    - Similarity-based (graded) entailment approach
    - Conditional logic approach

A (graded) similarity-based account of truthlikeness

Equip the set of possible worlds W with some kind of metric or, dually, similarity measure

Here, a $\otimes$-similarity relation on W is a mapping $S : W \times W \to [0,1]$
$S(w, w') :=$ how much similar is w to w'

- **Reflexivity**: $S(u, u) = 1$
  Separation: $S(u, v) = 1$ only if $u = v$
- **Symmetry**: $S(u, v) = S(v, u)$
- **$\otimes$-Transitivity**: $S(u, v) \otimes S(v, w) \leq S(u, w)$

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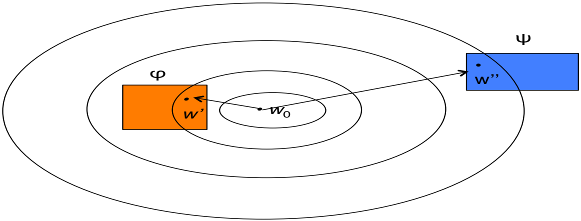
Weaker notions: closeness relations (Refl),
proximity, tolerance relations (Refl + Sim)

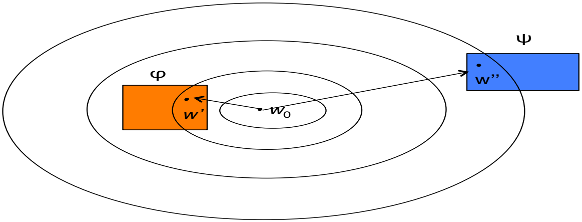
- a more informed scenario: complete information w_0 + precise concepts + a similarity S between possible worlds



Both φ and ψ are false at w_0 but

φ is closer to be true (more truthlike) than ψ

- a more informed scenario: complete information w_0 + precise concepts + a similarity S between possible worlds



Both φ and ψ are false at w_0 but

φ is closer to be true (more truthlike) than ψ

and now this can be quantified:

$$\mathit{truthlikeness}(\varphi) = \max\{S(w_0, w') \mid w' \models \varphi\} \geq \max\{S(w_0, w'') \mid w'' \models \psi\} = \mathit{truthlikeness}(\psi)$$

A more fine-grained representation and reasoning framework:

- In the enriched ideal scenario (w_0 + precise concepts + similarity) we still have the partition:

$$\mathbf{T} = \{\varphi \mid w_0 \models \varphi\} \quad \mathbf{F} = \{\psi \mid w_0 \models \neg\psi\}$$

but now we can refine it: $\mathbf{F} = \cup_{\alpha<1}$ α-Truthlike, where:

$$\alpha\text{-Truthlike} = \{\psi \mid \mathit{truthlikeness}(\psi) = \alpha\}$$

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$$\alpha\text{-Truthlike} = \{\psi \mid \textit{truthlikeness}(\psi) = \alpha\}$$

- More generally, given a theory (epistemic state), one may identify which consequences are closer (more truth-like) to hold than others

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      - approximate, strong, proximity entailments
    - Conditional logic approach

Focus: two kinds of entailment tolerant to small changes

Given $\varphi \models \psi$

1) How to define $\varphi \models^* \psi'$ such that:

If ψ' is similar to ψ, $\varphi \models \psi'$ still remains "valid"

- when φ is true, the more ψ' is similar to ψ, the more truth-like is ψ'

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Example:

scheduled departure time $= t \quad \models \quad$ flight departs at least at $t + 15min$

so

scheduled departure time $= t \quad \not\models \quad$ flight departs at $t + 10min$

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Focus: two kinds of entailment tolerant to small changes

Given $\varphi \models \psi$

2) How to define $\varphi \approx\!\!\!\models^* \psi$ such that:

If φ' is similar to φ, $\varphi' \models \psi$ still remains "valid"

- the less φ' is similar to φ, the stronger $\approx\!\!\!\models^*$ should be

Focus: two kinds of entailment tolerant to small changes

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2) How to define $\varphi \models\!\approx^* \psi$ such that:

If φ' is similar to φ, $\varphi' \models \psi$ still remains "valid"

- the less φ' is similar to φ, the stronger $\models\!\approx^*$ should be

Example:

current date is before the expiration date $\models$ *you can take the yoghourt*

Focus: two kinds of entailment tolerant to small changes

Given $\varphi \models \psi$

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Example:

current date is before the expiration date $\models$ *you can take the yoghourt*

but yoghourt producers want to be in the safe side, so:

current date is one day after the expiration date $\approx\!\!\!\models^*$ *you can take the yoghourt*

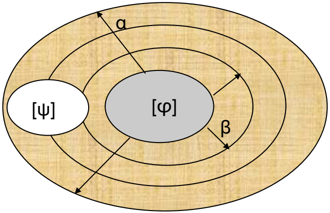
Approximate entailment

$S : W \times W \to [0,1]$

$\Rightarrow$ spheres around the set of models of a proposition $[\varphi]$

$$U_\alpha([\varphi]) = \{w \in W \mid \text{ exists } w' \in [\varphi] \text{ and } S(w', w) \geq \alpha\}$$

$$[\varphi] = U_1([\varphi]) \subseteq \ldots \subseteq U_\alpha([\varphi]) \subseteq \ldots \subseteq U_0([\varphi]) = W$$



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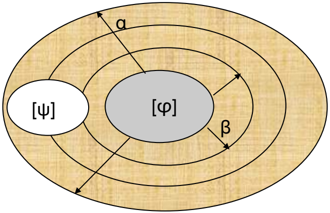

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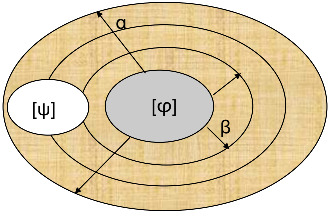

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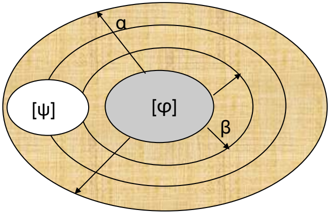

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Approximate entailment (cf. DEGGP,97): Given a $\otimes$-similarity $S : W \times W \to V$, with $V \subseteq [0,1]$, define:

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Characterizing properties:

(1)	**Supraclassicality:**	if $\varphi \models \psi$ then $\varphi \models^{\alpha} \varphi$ (in particular $\varphi \models^{1} \varphi$)
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From Approximate to Strong entailment

Approximate reasoning: derivation of approximate consequences

If φ then approximately ψ

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Strong reasoning: inferences tolerant to small changes in the premise

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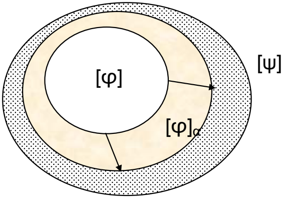

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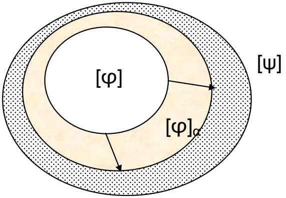

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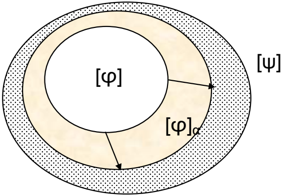

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Definition (EGRV, 2011): Given a $\otimes$-similarity relation $S : W \times W \to V$

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Yet another type: Proximity entailment

Approximate entailment: $\varphi \models_S^\alpha \psi$ holds iff $[\varphi] \subseteq U_\alpha([\psi])$

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"If approximately φ then approximately* ψ "

Compare: Approximate entailment: If φ then approximately ψ
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$\varphi \mathrel{|\!\equiv}^{\alpha}_{K} \psi$ and $\varphi \models^{\alpha}_{K} \psi$ are no longer equivalent !

Vagueness and Approximate consequences: "Heap" example

h_n: n grains of sand form a heap

Assumptions:

- 1000 grains of sand forms a heap: h_{1000} holds true
- If n grains of sand form a heap then $n-1$ grains form a heap as well: $h_n \rightarrow h_{n-1}$ holds true

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Sorites paradox:

$\{h_{1000}\} \cup \{h_n \to h_{n-1} : n \leq 1000\} \models h_1$

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Then we have:

$$h_n \models^{0.999}_{S,K} h_{n-1}$$

and

$$h_{1000} \models^{0.001}_{S,K} h_1$$

case-based reasoning

CBR: problem solving method in AI based on the principle that

"Similar problems have similar solutions"

Given a base of already solved problems (cases) and a new problem, the CBR cycle is:

1. RETRIEVE the most similar case(s)
2. REUSE the information and knowledge in that case(s) to solve the problem
3. REVISE the proposed solution
4. RETAIN the parts of this experience likely to be useful for future problem solving

Objects: described by a set $\mathcal{A}$ of attributes $\mathbf{d} = (a^1, \ldots a^r)$
Classes: $\mathcal{CL} = \{class^1, \ldots class^m\}$
$BC = \{(\mathbf{d_i}, class_i) \mid i = 1, \ldots n\}$: case-base of already classified objects
$\mathbf{d}^*$: new problem

$K =$ "The more similar is $\mathbf{d}^*$ to $\mathbf{d_i}$,
the more plausible $class_i$ is the class for d^*"

Given $\otimes$-similarities S_1 on $\mathcal{A}^n$ and S_2 on $\mathcal{CL}$ and let $S = S_1 \times S_2$. Then

$$\frac{\mathbf{d}^* \models^{\alpha}_{S} \mathbf{d_i} \qquad \mathbf{d_i} \models^{\beta}_{S,K} class_i}{\mathbf{d}^* \models^{\alpha\otimes\beta}_{S,K} class_i}$$

Assign to $\mathbf{d}^*$ the class which is an approximate consequence with highest degree.

Outline

- Introduction: uncertainty, fuzziness and truthlikeness
- Some logical approaches to reason under uncertainty
- Truthlikeness and similarity-based reasoning
  - Similarity-based (graded) entailment approach

    - approximate, strong, proximity entailments
  - Conditional logic approach

Logics of approximate and strong entailments

Aim: encode graded entailments "$\varphi \models^{\alpha}_{S} \psi$" and "$\varphi \approx^{\alpha}_{S} \psi$" as syntactic objects by conditional-like formulas

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Language(s):
- if φ, ψ are CPC propositions and $\alpha \in C \subset [0,1]$, then

$$\varphi >_\alpha \psi \qquad \varphi \succ_\alpha \psi$$

are LAE and LSE graded conditional formulas resp.

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- LAE language: built from conditionals $\varphi >_\alpha \psi$ and CPC connectives;

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(no nested conditional formulas !!)

- LASE language: analogously built with both kinds of conditionals

Semantics: Kripke-like models $M = (W, e, S)$, where:

- W set of possible worlds
- e : *Propositions* $\to 2^W$
- $S : W \times W \to V \subset [0, 1]$ is a $\otimes$-similarity

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$$M \models \varphi >_\alpha \psi \quad \text{if} \quad e(\varphi) \subseteq U_\alpha(e(\psi))$$

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- CPC formulas φ can be interpreted into LAE (resp. LSE) as $\top >_1 \varphi$ (resp. $\top \succ_1 \varphi$)

LAE fragment: a logic of approximate entailment

Axioms and Rule:

(A1) $\phi >_1 \psi$, if $\phi \to \psi$ is a tautology of CPL

(A2) $(\phi >_\alpha \psi) \to (\phi >_\beta \psi)$, where $\alpha \geq \beta$

(A3) $(\phi >_0 \psi) \vee (\psi >_1 \bot)$

(A4) $(\phi >_\alpha \bot) \to (\phi >_1 \bot)$

(A5) $(\delta >_\alpha \epsilon) \ \to \ (\epsilon >_\alpha \delta) \vee (\delta >_1 \bot)$, where δ, ϵ are m.e.c.'s

(A6) $(\phi >_\alpha \chi) \wedge (\psi >_\alpha \chi) \ \to \ (\phi \vee \psi >_\alpha \chi)$

(A7) $(\epsilon >_\alpha \phi \vee \psi) \ \to \ (\epsilon >_\alpha \phi) \vee (\epsilon >_\alpha \psi)$, where ϵ is a m.e.c.

(A8) $(\phi >_1 \psi) \to (\phi \wedge \neg\psi >_1 \bot)$

(A9) $(\phi >_\alpha \psi) \wedge (\psi >_\beta \chi) \to (\phi >_{\alpha \odot \beta} \chi)$

(A10) LAE-formulas obtained by uniform replacements of variables in CPL-tautologies by LAE graded conditionals

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Completeness: $T \vdash_{LAE} \Phi$ iff $T \models_{LAE} \Phi$

LSE fragment: a logic of strong entailment

Axioms and Rule:

(S1) $\phi \succ_1 \psi$, if $\phi \to \psi$ is a tautology of CPL

(S2) $\bot \succ_0 \phi, \qquad \phi \succ_0 \top$

(S4) $(\phi \succ_0 \psi) \ \to \ (\phi \succ_1 \bot) \vee (\top \succ_1 \psi)$

(S5) $(\phi \succ_\alpha \psi) \to (\phi \succ_\beta \psi)$, where $\alpha \leq \beta$

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(S9) $(\phi \succ_\alpha \psi) \wedge (\psi \succ_\beta \chi) \ \to \ (\phi \succ_{\min\{\alpha,\beta\}} \chi)$

(S10) $(\phi \succ_{\alpha\odot\beta} \psi) \ \to \ (\epsilon \succ_\alpha \neg\phi) \vee (\epsilon \succ_\beta \psi)$, where ϵ is a m.e.c.

(A10) LSE-formulas obtained by uniform replacements of variables in CPL-tautologies by LSE graded conditionals

(MP) Modus Ponens

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Axioms and Rule:

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(A10) LSE-formulas obtained by uniform replacements of variables in CPL-tautologies by LSE graded conditionals

(MP) Modus Ponens

Completeness: $\mathcal{T} \vdash_{LSE} \Phi$ iff $\mathcal{T} \models_{LSE} \Phi$

LASE: merging LAE and LSE

Axioms and Rule:

(AS0) Axioms of LAE and LSE

(AS1) $(\phi >_1 \psi) \leftrightarrow (\phi \succ_1 \psi)$

(AS2) $(\phi >_\alpha \psi) \wedge (\psi \succ_\alpha \chi) \rightarrow (\phi >_1 \chi)$

(AS3) $(\epsilon >_\alpha \delta) \leftrightarrow \neg(\delta \succ_\alpha \neg\epsilon)$, where ϵ, δ are m.e.c.'s

(AS4) Given a tautology of CPL, the statement resulting from a uniform replacement of the atoms by graded LAE-implications or graded LSE-implications is an axiom.

(MP) Modus Ponens

LASE: merging LAE and LSE

Axioms and Rule:

- (AS0) Axioms of LAE and LSE
- (AS1) $(\phi >_1 \psi) \leftrightarrow (\phi \succ_1 \psi)$
- (AS2) $(\phi >_\alpha \psi) \wedge (\psi \succ_\alpha \chi) \rightarrow (\phi >_1 \chi)$
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- (AS4) Given a tautology of CPL, the statement resulting from a uniform replacement of the atoms by graded LAE-implications or graded LSE-implications is an axiom.
- (MP) Modus Ponens

Completeness: $\mathcal{T} \vdash_{LASE} \Phi$ iff $\mathcal{T} \models_{LASE} \Phi$

Conclusions

- A number of different logical approaches to formalize reasoning under uncertainty, fuzziness and truthlikeness
- Different kinds of languages and levels of expressivity
- Many issues not addressed here:
  - similarity-based modal logics
  - computational complexity issues
  - etc .
- To be further explored: combinations of uncertainty / fuzziness / truthlikeness
  - preliminary results alrerady available for possibilistic, probabilistic and belief functions models
- Introducing rough sets into the picture

Thank you !

On complexity results of fuzzy probability logics

(Hájek-Tulipani, FI 2001), (Hájek, FSS 2007)

General logics $FP(L_1, L_2)$ where L_1 is Boolean logic or a t-norm fuzzy logic and L_2 a t-norm fuzzy logic

- $Sat(FP(CPC, \text{Ł}))$ is NP-complete, $Taut(FP(CPC, \text{Ł}))$ is co-NP-complete

  $Sat(FP(CPC, \text{Ł}\Pi))$ and $Taut(FP(CPC, \text{Ł}\Pi))$ are in PSPACE

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- $Sat(FP(Ł_n, Ł))$ and $Sat(FP(G_n, Ł))$ are NP-complete

- $Sat(FP(G, L_2))$ and $Taut(FP(G, L_2))$ are in PSPACE, for L_2 being an arbitrary suitable logic

  $Sat(FP(L_1, Ł))$ and $Taut(FP(L_1, Ł))$ are in PSPACE, for L_1 being an arbitrary suitable logic

LPE: logic of proximity entailment

Axioms:

CPC:	tautologies of CPC
N:	$\varphi \gg_\alpha \psi \rightarrow \varphi \gg_\beta \psi$ if $\beta \leq \alpha$
CS:	$\varphi \gg_1 \psi \rightarrow (\varphi \rightarrow \psi)$
EX:	$\varphi \gg_0 \psi$
4:	$(\varphi \gg_\alpha \psi) \wedge (\psi \gg_\beta \chi) \rightarrow \varphi \gg_{\alpha \otimes \beta} \chi$
LO:	$(\varphi \vee \psi \gg_\alpha \chi) \leftrightarrow (\varphi \gg_\alpha \chi) \wedge (\psi \gg_\alpha \psi)$
RO:	$(\chi \gg_\alpha \varphi \vee \psi) \leftrightarrow (\chi \gg_\alpha \varphi) \vee (\chi \gg_\alpha \psi)$

Rules:

From $\varphi \rightarrow \psi$ infer $\varphi \gg_1 \psi$

Completeness (Rodriguez, 2002): if T finite, $T \vdash_{LPE} \Phi$ iff $T \models_{LPE} \Phi$

Related papers

- P. Hájek, L. Godo, F. Esteva (1995) *Probability and Fuzzy Logic*. In Proc. of UAI'95, 237–244.

- F. Esteva, L. Godo, P. Hájek (2000) *Reasoning about probability using fuzzy logic*. Neural Network World 10, No. 5, 811–824.

- L. Godo, P. Hjek, F. Esteva (2001) *A fuzzy modal logic for belief functions*. Proc. IJCAI'01, 723-729.

- T. Flaminio, L. Godo (2007) *A logic for reasoning about the probability of fuzzy events*. Fuzzy Sets and Systems 158, 625 - 638.

- T. Flaminio, L. Godo, E. Marchioni (2008) *On the Logical Formalization of Possibilistic Counterparts of States over n-valued Łukasiewicz Events*. JLC, 2011.

- F. Esteva, L. Godo, R. Rodriguez, T. Vetterlein *Logics for approximate and strong entailments*. FSS, 2012.

- T. Flaminio, L. Godo, E. Marchioni (2012) *Logics for belief functions on MV-algebras*. IJAR, to appear.