Representations of MV-algebras and consequences

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Representation of MV-algebras as subdirect product

Theorem

Every MV-algebra is a subdirect product of MV-chains.

Examples of MV-chains

The standard MV-algebra $([0,1], \oplus, \neg, 0)$
where $x \oplus y = \min\{x+y, 1\}$ and $\neg x = 1 - x$, for any $x, y \in [0,1]$.

The Chang's MV-algebra $C = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1,0))$.

The nonstandard MV-algebra $({}^*[0,1], \oplus, \neg, 0)$
where an element of ${}^*[0,1]$ is a nonstandard real between ^{*0} and ^{*1}.

Representation of MV-algebras as algebras of functions

Theorem

Any semisimple MV-algebra is a subdirect product of subalgebras of $[0,1]$. In other words, the semisimple MV-algebras are exactly the bold algebras of fuzzy sets.

Theorem

Each MV-algebra A is, up to isomorphisms, a subalgebra of the algebra of functions valued in $[0,1]^*$ and defined on an opportune set X, being $[0,1]^*$ an ultrapower of $[0,1]$ which only depends on the cardinality of A.

Theorem

Each MV-algebra with limitated cardinality embeds in a fixed MV-algebra of functions valued in a not standard ultrapower of $[0,1]$.

Representation of MV-algebras by sheaves

Let A be an MV-algebra. For each $M \in \mathrm{Max}(A)$, consider

$$O(M) = \bigcap \{m \in \mathrm{Min}(A) \mid m \subseteq M\}.$$

$O(M)$ is a primary ideal and, so, $A/O(M)$ is a local MV-algebra.

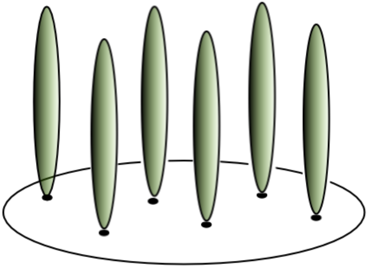
$$\bigcap \{O(M) \mid M \in \mathrm{Max}(A)\} = \{0\}.$$

Let $\mathcal{F} = (\mathrm{Max}(A), \pi, E)$ be a sheaf such that E is the disjoint union of the quotients $A/O(M)$ for each $M \in \mathrm{Max}(A)$ and $\pi : E \to \mathrm{Max}(A)$ is a local homeomorphism such that $\pi^{-1}(M) = A/O(M)$.

Representation of MV-algebras by sheaves

Theorem

Every MV-algebra is isomorphic to the MV-algebra of all global sections of the sheaf $\mathcal{F}$.



Local MV-algebras

Proposition

For any MV-algebra A, the following are equivalent:

(*a*) for any $x \in A$, $\mathrm{ord}(x) < \infty$ or $\mathrm{ord}(\neg x) < \infty$,

(*b*) for any $x, y \in A$, $x \odot y = 0$ implies $x^n = 0$ or $y^n = 0$ for some $n \in \omega$,

(*c*) for any $x, y \in A$, $\mathrm{ord}(x \oplus y) < \infty$ implies $\mathrm{ord}(x) < \infty$ or $\mathrm{ord}(y) < \infty$,

(*d*) $\{x \in A : \mathrm{ord}(x) = \infty\}$ is a proper ideal of A,

(*e*) A has only one maximal ideal,

(*f*) for any $x \in A$, there is an integer $n \geq 1$ such that $(nx)^2 \in \{0, 1\}$,

(*g*) $\mathrm{Rad}(A)$ is a prime ideal.

An MV-algebra A is called *local* if one of the previous equivalent conditions holds.

Local MV-algebras

Theorem

The class of local MV-algebras is a universal class. Indeed an MV-algebra A is local iff for each $x \in A$, $x \leq \neg x$, $\neg x \leq x$ or $(d(x, \neg x))^2 = 0$.

Theorem

Let A be an MV-algebra and P an ideal of A. Then A/P is local iff P is primary,

where P is primary iff if $x \odot y \in P$ then $x^n \in P$ or $y^n \in P$ for some $n \in \omega$.

An example of local MV-algebras

Now we give an example of local MV-algebras which is a kind of prototypical local MV-algebra. Indeed, let X be an arbitrary nonempty set, U an MV-algebra, and $\mathbf{K}(U^X)$ the subset of the MV-algebra U^X as follows:

$$\mathbf{K}(U^X) = \{f \in U^X \mid f(X) \subseteq [a]_{\mathrm{Rad}(U)} \quad \text{for some} \quad a \in U\}.$$

$\mathbf{K}(U^X)$ shall be called the *the full MV-algebra of quasi constant functions* from X to U. Of course any element f from $\mathbf{K}(U^X)$ shall be said *quasi constant* function from X to U. Any subalgebra of $\mathbf{K}(U^X)$ shall be called an algebra of quasi constant functions.

$\mathbf{K}(U^X)$ is a local MV-algebra.

Theorem

Every local MV-algebra can be embedded into an MV-algebra of quasi costant functions.

Perfect MV-algebras

Definition

An MV-algebra A is called *perfect* if for every nonzero element $x \in A$, $\mathrm{ord}(x) = \infty$ iff $\mathrm{ord}(\neg x) < \infty$.

Theorem

The class of perfect MV-algebra is universal. Indeed, an MV-algebra A is perfect iff A satisfies $\sigma \& \tau$, where σ is the wff

$$(\forall x)(x^2 \oplus x^2 = (x \oplus x)^2)$$

and τ is the wff

$$(\forall x)(x^2 = x \Rightarrow (x = 0 \text{ OR } x = 1)).$$

Perfect MV-algebras

Definition

A proper ideal P of an MV-algebra A is called *perfect* iff for every $a \in A$, $a^n \in P$ for some $n \in \omega$ iff $(\neg a)^m \notin P$ for all $m \in \omega$.

Theorem

Let A be an MV-algebra and P an ideal of A. Then A/P is perfect iff P is perfect.

An example of perfect MV-algebra is the Chang's MV-algebra C.

In what follows, **Perfect**, **Local**, $V(C)$ and $V($**Perfect**$)$ will respectively indicate the classes of local and perfect MV-algebras and the varieties generated by C and all perfect MV-algebras.

Some pertinent facts about perfect MV-algebras

Proposition

The following hold:

1. The only finite perfect MV-algebra is $\{0, 1\}$.
2. Every nonzero element in a perfect MV-algebra $A \neq B(A)$ generates a subalgebra isomorphic to the Chang algebra C.
3. Subdirect irreducible algebras in $V(C)$ are all perfect MV-chains.
4. $V(\textbf{Perfect}) = V(C)$.
5. **Perfect**$= V(C) \cap$**Local**.
6. A is perfect iff $A = \langle \mathrm{Rad}(A) \rangle = \mathrm{Rad}(A) \cup \neg \mathrm{Rad}(A)$
7. $x \in \mathrm{Rad}(A)$ iff $\mathrm{ord}(x) = \infty$.
8. **Perfect** is closed under homomorphic images and subalgebras.
9. A is perfect iff any proper ideal of A is perfect.
10. A is perfect iff $\{0\}$ is a perfect ideal.

Localization

Let A be an MV-algebra and P a prime ideal of A. Let

$$\mathcal{L}(P) = \{A' \mid A' \text{ is a subalgebra of } A \text{ and } P \text{ is maximal in } A'\}$$

and

$$\omega_P(A) = \{Q \in \mathrm{Spec}(A) \mid Q \subseteq P\}.$$

Theorem

Let A be an MV-algebra and $P \in \mathrm{Spec}(A)$. For any subalgebra $A' \in \mathcal{L}(P)$, $\mathrm{Spec}(A'/O(P))$ is homeomorphic to a subspace of $\mathrm{Spec}(A)$. In particular $\mathrm{Spec}(A'/O(P))$ is homeomorphic to $\omega_P(A)$.

The Boolean and the perfect skeletons of an MV-algebra

Theorem

For each MV-algebra A,

$$B(A) = \{x \in A \mid x \oplus x = x\}$$

is the greatest Boolean subalgebra of A, that is $B(A)$ is the Boolean skeleton of A.

Theorem

For each MV-algebra A,

$$P(A) = \mathsf{Rad}(A) \cup \neg\,\mathsf{Rad}(A)$$

is the greatest perfect subalgebra of A, that is $P(A)$ is the perfect skeleton of A.

The local skeleton of an MV-algebra

Theorem

For each MV-algebra A,

$$\mathcal{L}(A) = \left\{ x \in A \mid \text{ for every } P \in \mathrm{Spec}(A), \frac{x/P}{\mathrm{Rad}(A/P)} = r_x \in [0,1] \right\}$$

is the greatest local subalgebra of A, that is $\mathcal{L}(A)$ is the local skeleton of A.

The $V(C)$-skeletons of an MV-algebra

Let $V(C)$ the variety generated by the Chang's MV-algebra C.

Theorem

Let A be an MV-algebra. The following are equivalent

1. $A \in V(C)$.
2. $A/\operatorname{Rad}(A) \cong B(A)$.
3. for $x \in A$, $2x^2 = (2x)^2$.[1]

Theorem

For each MV-algebra A,

$$A_0 = \bigcap_{M \in \operatorname{Max}(A)} (M \cup \neg M) = \langle B(A) \cup \operatorname{Rad}(A) \rangle$$

is the greatest subalgebra of A such that $A_0 \in V(C)$, that is A_0 is the $V(C)$-skeleton of A.

[1] Remember Cignoli-Torrens' DL-algebras.

The $V(S_n)$-skeleton of an MV-algebra

For any $n \in \omega$, we can define the following MV-algebras.

$$S_n = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$$

Lemma

Let A be an MV-chain. Then for $n \in \omega$ there exists the greatest subalgebra $F_n(A)$ of A such that $F_n(A) \in V(S_n)$.

Theorem

Let A be an MV-algebra and consider A as subdirect product of a family of MV-chains $\{A_i\}_{i \in I}$. Then for $n \in \omega$,

$$W_n(A) = \{x \in A \mid x_i \in F_n(A_i) \text{ for every } i \in I\}$$

is the greatest subalgebra of A, such that $W_n(A) \in V(S_n)$, that is $W_n(A)$ is the $V(S_n)$-skeleton of A.

The $V(S_n^\omega)$-skeleton of an MV-algebra

For any $n \in \omega$, we can define the following MV-algebras.

$$S_n^\omega = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (n, 0))$$

Theorem

Let A be an MV-algebra. Then for $n \in \omega$,

$$W_n^\omega(A) = \{x \in A \mid x/\operatorname{Rad}(A) \in F_n(A/\operatorname{Rad}(A)\}$$

is the greatest subalgebra of A, such that $W_n^\omega(A) \in V(S_n^\omega)$, that is $W_n^\omega(A)$ is the $V(S_n^\omega)$-skeleton of A.

Local MV-algebras with retractive radical

Definition

An ideal H of an MV-algebra A is called *retractive* iff there is a homomorphism $h : A/H \to A$ such that $\pi_H \circ h$ is the identity map of A/H, where π_H is the canonical projection from A to A/H.

Theorem

Let A be a local MV-algebra. Then the following are equivalent:

(i) there is a subgroup $\mathbb{R}'$ of $\mathbb{R}$ and an abelian ℓ-group G such that

$$A \cong \Gamma(\mathbb{R}' \overrightarrow{\times} G, (1, 0))$$

(ii) A is radical retractive.

(iii) A is the coproduct of $A/Rad(A)$ and $\langle \mathrm{Rad}(A) \rangle$, in symbols $A = A/\,\mathrm{Rad}(A) \amalg \langle Rad(A) \rangle$.

Local MV-algebras with retractive radical

Let A be an MV-algebra. For any $X \subseteq A$ the set

$$Cl_A(X) = \{(x \oplus \varepsilon) \odot \neg\tau \mid x \in X, \, \epsilon, \tau \in \mathrm{Rad}(A)\}$$

is the set of clouds of infinitesimals around elements of X.

Proposition

Let A be a local MV-algebra and S be a simple subalgebra of A. Then $Cl_A(S)$ is a subalgebra of A (hence it is local) and is the coproduct of S and $\langle Rad(A) \rangle$:

$$Cl_A(S) \cong S \amalg \langle Rad(A) \rangle.$$

Local MV-algebras with retractive radical

Let A be an MV-algebra. An element x of A is called *finite* iff $\mathrm{ord}(x) < \infty$ and $\mathrm{ord}(\neg x) < \infty$. Let $Fin(A)$ denote the set of all finite elements of A.

Theorem

Let A be an MV-algebra. Then the following are equivalent:

(i) A is local and $Fin(A) \cup \{0\}$ is a subalgebra of A;

(ii) there is a subgroup $\mathbb{R}'$ of $\mathbb{R}$ and an ℓ-group G such that

$$A = \Gamma\left(\left(\mathbb{R}' \overrightarrow{\times} G\right), (1, 0)\right).$$

In this case, $Fin(A) \cup \{0\}$ is the biggest simple subalgebra of A.

Local MV-algebras with retractive radical

So local MV-algebras in which the finite elements together with 0 form a subalgebra, are coproducts of a simple MV-algebra and a perfect one. Further, they can be described by considering the unit interval of a subgroup $\mathbb{R}'$ of $\mathbb{R}$ and adding clouds of infinitesimals for each $r \in \mathbb{R}'$.

But this is a very special case. In general such representation cannot be used for any local MV-algebra. Think for example of $\Gamma(\mathbb{R} \overrightarrow{\times} G, (1,1))$.

Theorem

Every local MV-algebra can be embedded into a local MV-algebra whose radical is retractive.

Local MV-algebras of finite rank

Definition

Let n be a positive integer. Then a local MV-algebra A is said to be of *rank* n iff $A/Rad(A) \cong S_n$. A local MV-algebra A is said to be of *finite rank* iff A is of rank n for some integer n.

Let $\textsc{FinRank}$ denote the class of all local MV-algebras of finite rank and $\textsc{FinRank}_n$ denote the class of local MV-algebras of rank n.

Proposition

Let A be a local MV-algebra. Then the following are equivalent:

(i) $A \in \textsc{FinRank}$ and A is an MV-chain;

(ii) $A \cong \Gamma(Z \overrightarrow{\times} G, (n, g))$ for some integer n, with G a totally ordered abelian group and g element of the positive cone of G.

Local MV-algebras of finite rank

Theorem

Let A be an MV-algebra. Then the following are equivalent:

(i) $A \in \textsc{FinRank}_n$;

(ii) $A \cong \Gamma(Z \overrightarrow{\times} G, (n, g))$ with G abelian l-group and $g \in G$.

Theorem

Every local MV-algebra of rank n can be embedded into a local MV-algebra of rank n whose radical is retractive.

Indeed, $A \cong \Gamma(Z \overrightarrow{\times} G, (n, g))$ for some integer n, with G abelian l-group and $g \in G$. It is not difficult to prove that A embeds in $B = \Gamma(Z \overrightarrow{\times} G, (n, 0))$ which is a local algebra of rank n whose radical is retractive.

Where local MV-algebras meet varieties

Definition

Let I, J be subsets of $\mathbb{N}$. We denote by $\textsc{FinRank}(I, J)$ the class of local MV-algebras A of finite rank such that either A is simple and $A \cong A/\operatorname{Rad}(A)$ is embeddable into a member of $\{S_i \mid i \in I\}$, or A is not simple and $A/\operatorname{Rad}(A)$ is embeddable into a member of $\{S_j \mid j \in J\}$.

Theorem

$\textsc{Loc}(V(S_{n_1}, ..., S_{n_h}; S^{\omega}_{m_1}, ..., S^{\omega}_{m_k})) =$
$\textsc{FinRank}(\{n_1, ..., n_h\}, \{m_1, ..., m_k\})$.

Theorem

The class $\textsc{Loc}(V(S_{n_1}, ..., S_{n_h}; S^{\omega}_{m_1}, ..., S^{\omega}_{m_k}))$ is universal.

Where local MV-algebras meet varieties

Theorem

Let A be an MV-algebra and n a positive integer. The following statements are equivalents:

1) $A \in V(S_n^\omega)$ and A satisfies the following formula

$$\tau_n : (\forall x)\left((2x = 1) \vee \left(x^2 = 0\right) \vee \left((n+1)\,x = 1 \wedge \left(x^{n+1} = 0\right)\right)\right),$$

2) A is a local MV-algebra of *rank* d, with d divisor of n.

Where local MV-algebras meet varieties

Theorem
For a given arbitrary nonsimple MV-chain A of rank n it is:

$$ISP_u(A) = ISP_u(\Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (n, d_A))),$$

where d_A is the maximum m such that S_m is embeddable in A.

Theorem
Let A be an MV-algebra and n a positive integer. Then the following statements are equivalent:

1) $A \in ISP_U(\mathrm{Loc}_{n,0})$.
2) $A \in \mathrm{Loc}(V(S_n^\omega))$.

Weak Boolean Products

Theorem
All MV-algebras are weak Boolean products of indecomposable MV-algebras.

Theorem
Weak Boolean products of MV-chains are, up to isomorphisms, just those algebras A having the lattice reduct $L(A)$ that is a dual Stone algebra (*dual Stone* MV-algebras).

Theorem
Weak Boolean products of simple MV-chains are exactly the hyperarchimedean MV-algebras.
Weak Boolean products of finite MV-chains are the *liminary* MV-algebras, that is MV-algebras whose quotients by prime ideals are finite MV-chains

Weak Boolean Products

Definitions

An MV-algebra A is *quasi local* iff for any $x \in A$ there exist $b \in B(A)$ and $n \geq 1$ such that $nx \oplus b = 1$ and $n(x^*) \oplus b^* \neq 1$.

An MV-algebra A is *quasi perfect* iff it is quasi local and for any $x \in A$ and $b \in B(A) \setminus \{1\}$ if $(x \oplus b) \notin M$, for every $M \in Max(A)$ then there exists $M' \in Max(A)$ such that $\neg x \oplus b \in M$.

Theorem

Weak Boolean products of local MV-algebras are *quasi local* MV-algebras.
Weal Boolean products of perfect MV-algebras are *quasi perfect* MV-algebras.

Weak Boolean Products

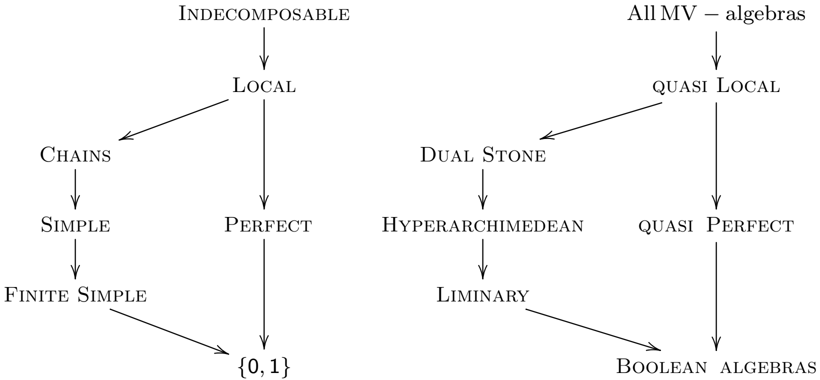


Figure: Classes of MV-algebras and correspondent weak boolean products

Weak Boolean Products

Definition

An MV-algebra A is called *quasi local of finite rank* iff it is quasi local and there exists an integer n such that for all $a \in A$ and $b \in B(A) \setminus \{1\}$ if for every $M \in Max(A)$, $(a \oplus b) \notin M$ then there exists $M_0 \in Max(A)$ such that $(a^*)^n \oplus b \in M_0$.

Theorem

Weak Boolean products are quasi local MV-algebras of finite rank.

Theorem

Let A be an MV-algebra and V any proper subvariety of MV. Then the following are equivalent:

(i) $A \in V$;

(ii) $A \in WBP(\mathrm{Loc}(V))$.

Weak Boolean Products

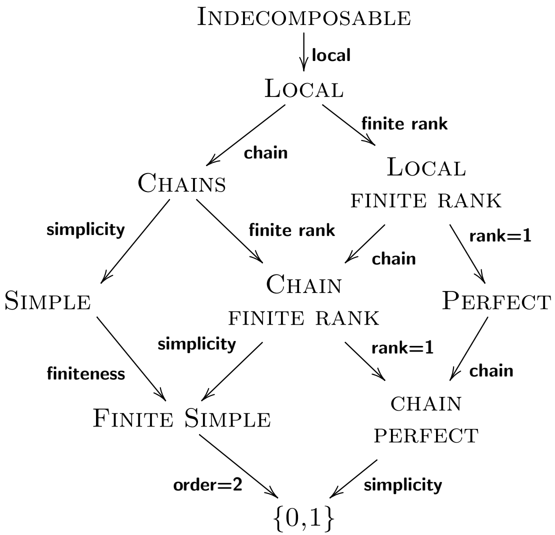


Figure: Class of algebras as factors of weak boolean products

Weak Boolean Products

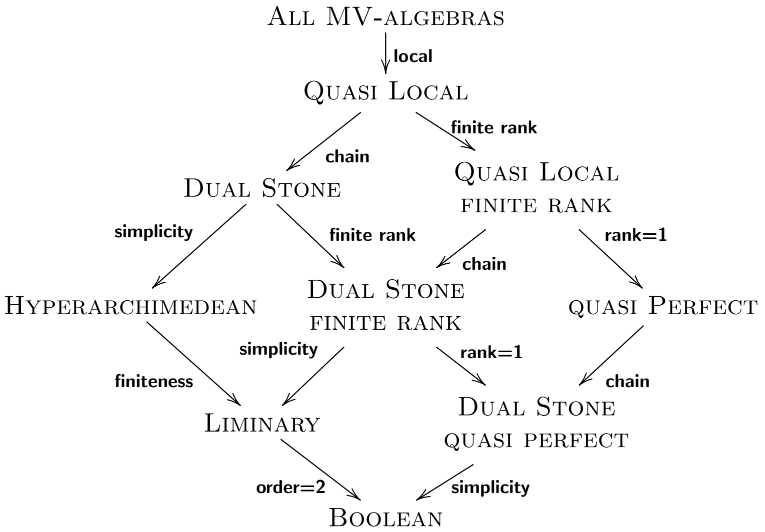


Figure: Class of algebras generated by classes in Figure 2 by means of Weak Boolean product

Propositional Logic of perfect MV-algebras

Let **Luk** denote the Łukasiewicz propositional logic with usual axioms and rule of inference.

Let $\mathbf{Luk}_P$ be the axiomatic extension of **Luk** by adding the following axiom

$$(2\alpha^2) \leftrightarrow (2\alpha)^2.$$

Proposition

A wff of $\mathbf{Luk}_P$ is valid on all perfect MV-chains iff it is provable in $\mathbf{Luk}_P$.

Propositional Logic of perfect MV-algebras

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Let $\mathbf{Luk}_P$ be the axiomatic extension of **Luk** by adding the following axiom

$$(2\alpha^2) \leftrightarrow (2\alpha)^2.$$

Proposition

A wff of $\mathbf{Luk}_P$ is valid on all perfect MV-chains iff it is provable in $\mathbf{Luk}_P$.

First order Logic of perfect MV-algebras

Let **L** denote the first order Łukasiewicz logic with usual axioms and rule of inference.

Let ˜**Luk**$_P$ be the axiomatic extension of **L** by adding the following axiom

$$(2\alpha^2) \leftrightarrow (2\alpha)^2.$$

Proposition

A wff of ˜**Luk**$_P$ is valid on all perfect MV-chains iff it is provable in ˜**Luk**$_P$.

Presheaf of local MV-algebras

Let A be an MV-algebra and Spec(A) the spectrum of prime ideals of A endowed with the Zariski topology.

Let $\mathbf{O}(\mathrm{Spec}(A))$ be the category of open sets of Spec(A) having a morphism from U to V iff $U \subseteq V$.

For each open set U of Spec(A), let $A_U = \mathcal{L}(\prod_{P \in U} A/P)$.

Theorem

The map $\mathcal{F} : \mathbf{O}(\mathrm{Spec}(A))^{op} \to \mathbf{Loc}$ defined as $\mathcal{F}(U) = A_U$ for each $U \in \mathrm{Ob}(\mathbf{O}(\mathrm{Spec}(A))^{op})$ is a presheaf of local MV-algebras.

Presheaf of perfect MV-algebras

Let $\{A_i\}_{i \in I}$ be a family of MV-algebras and let $A = \prod_{i \in I} A_i$. By *pseudo-diagonal* of A is meant the set of all $a \in A$ such that $\mathrm{ord}(a_i) = \mathrm{ord}(a_j)$ for all $i, j \in I$. We will indicate the pseudo-diagonal of A by $p\delta_{i \in I} A_i$.

Proposition

The pseudo-diagonal of a family of perfect MV-algebras is a perfect MV-algebra.

Proposition

Let A be an MV-algebra. The MV-algebra $A_P = \langle P \rangle / O(P)$ is the unique perfect MV-algebra with maximal ideal $P/O(P)$.

Presheaf of perfect MV-algebras

Let $\mathcal{P}$ be the full subcategory of the category of MV-algebras whose objects are perfect MV-algebras.

Let A be an MV-algebra and $\mathbf{O}(\mathrm{Spec}(A))$ be the category of open sets of $\mathrm{Spec}(A)$ as before.

For each $U \in \mathrm{Ob}(\mathbf{O}(\mathrm{Spec}(A)))$, let $A_U = p\delta_{P \in U} A_P$.

Theorem

The map $\mathcal{F} : \mathbf{O}(\mathrm{Spec}(A))^{op} \to \mathcal{P}$ defined as $\mathcal{F}(U) = A_U$ for each $U \in \mathrm{Ob}(\mathbf{O}(\mathrm{Spec}(A))^{op})$ is a presheaf of perfect MV-algebras.