# Basic algebraic geometry on MV- algebras

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## Introduction

The spirit of this paper can be summarized by the following quotation, taken from Lawere:

*In spite of its geometric origin, topos theory has in recent years some-times been perceived as a branch of logic, partly because of the contributions to the clarification of logic and set theory which it has made possible. However, the orientation of many topos theorists could perhaps be more accurately summarized by the observation that what is usually called mathematical logic can be viewed as a branch of algebraic geometry, and it is useful to make this branch explicit in itself.*

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On the other hand, there are reasons to be interested in other MV algebras, because every MV algebra can be viewed as the Lindenbaum algebra of some many-valued logic, and as such, it has logical relevance.

This is why we try to generalize somewhat the theory of (M11) to MV algebras as general as possible.

In order to develop our theory we proceed along lines of algebraic geometry over varieties in universal algebra.

We note that in algebraic geometry the central notion is the one of polynomial. One has three possibilities:

- considering coefficient-free algebraic geometry; this allows one to evaluate polynomials in arbitrary fields;
- considering Diophantine algebraic geometry: this means that the field where coefficients are taken coincides with the field where polynomials are evaluated;
- considering general, non-Diophantine algebraic geometry, where polynomials take coefficients in a field $K$ and are evaluated in an extension $L$ of $K$.

It turns out that all these three possibilities can be extended to universal algebra.

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Since universal algebra subsumes the equational theory of MV algebras, we can consider what happens in universal algebraic geometry

- coefficient-free,
- Diophantine
- non-Diophantine

over MV algebras.

Because of the completeness theorem, we can say that all information for this connection is already provided by the MV algebra $[0, 1]$.

However, since we are interested in a Diophantine and Non-Diophantine approach to MV algebraic geometry, we would like to go beyond $[0, 1]$ and consider an MV algebra $A$.

This corresponds to adding to Łukasiewicz logic the atomic diagram of $A$.

Of course in the generalization we lose something

However, many concepts of the theory of McNaughton functions still make sense, like

- the category of algebraic sets and Z-maps (here replaced by polynomial maps) and
- the category of MV algebras and homomorphisms, as well as
- the equivalence between them.

**Term algebras**:

Let $X$ be a non-empty set of elements called variables, let $F$ be a type of algebra.
The set of *terms over* $X$ and $F$, denoted $\mathbf{T}(X, F)$, is the least set of strings of symbols such that $X \subseteq \mathbf{T}(X)$, if $t_1, \ldots, t_n \in \mathbf{T}(X)$ and $f \in F$ has arity $n$ then then $f(t_1, \ldots, t_n) \in \mathbf{T}(X)$.
$\mathbf{T}(X, F)$ is called a **term algebra**.

**MV − algebras and polynomials**

Let $A$ be an MV algebra and $n$ be a positive integer. Let $F_A$ be the language of MV algebras plus a constant symbol $c_a$ for every $a \in A$.

Define $A[x_1, \ldots, x_n]$ (the MV algebra of polynomials in $n$ variables with constants in $A$) to be the quotient $\mathbf{T}_n(X, F_A)/C_A$,

where $C_A$ is the congruence generated by the axioms for MV-algebras and the complete diagram of $A$.

**MV – Polynomial functions**:

In MV algebras (and in universal algebra in general) it is crucial to distinguish polynomials and polynomial functions.

Equal polynomials induce the same function everywhere,

but two polynomials can induce the same function on some MV-algebra without being equal.

Given an MV algebra $A$ and an MV term $p(x_1, \ldots, x_n) \in \mathbf{T}_n(X, A)$ we may define a function $p_A : A^n \to A$ as follows:
1) if $p(x_1, \ldots, x_n) = x_i$, then $p_A(a_1, \ldots, a_n) = a_i$;
2) if $p(x_1, \ldots, x_n) = c_a$ for some $a \in A$, then $p_A(a_1, \ldots, a_n) = a$;
3) if $p(x_1, \ldots, x_n) = p_1(x_1, \ldots, x_n) \oplus p_2(x_1, \ldots, x_n)$, then

$$p_A(a_1, \ldots, a_n) = p_{1A}(a_1, \ldots, a_n) \oplus_A p_{2A}(a_1, \ldots, a_n);$$

4) if $p(x_1, \ldots, x_n) = p_1(x_1, \ldots, x_n) \odot p_2(x_1, \ldots, x_n)$, then

$$p_A(a_1, \ldots, a_n) = p_{1A}(a_1, \ldots, a_n) \odot_A p_{2A}(a_1, \ldots, a_n);$$

5) $p^*_A(a_1, \ldots, a_n) = (p_A(a_1, \ldots, a_n))^{*_A}$.

We call $p_A$ the $\mathbf{MV-polynomial-function}$ induced on $A$ by $p$.

We note that polynomial functions on an MV algebra $A$ form an MV algebra, which we will denote by $PF_n(A)$. We write $p \equiv_A q$ if $p_a = q_A$.

If two terms over an algebra $A$ are the same polynomial, then they induce the same function on $A$, but not conversely. However we have the following characterization:

## Proposition

*Two terms $p, q$ in $n$ variables give the same polynomial on an MV-algebra $A$*

*if and only if*

*there is an extension $A'$ of $A[x_1, \ldots, x_n]$ such that $p, q$ are congruent modulo $\equiv_{A'}$.*

**McNaughton functions and McNaughton Theorem**:

We recall the notion of McNaughton functions and McNaughton Theorem.

A function $f$ from $[0,1]^n$ to $[0,1]$ is called a McNaughton function if it is continuous and there are $k$ linear polynomials with integer coefficients such that for every $y \in [0,1]^n$ there is $j$ such that $f(x) = p_j(x)$.

Then McNaughton Theorem says that McNaughton functions form an MV-algebra isomorphic to the free MV-algebra on $n$ generators.

**Truncated functions and a generalized McNaughton Theorem**:

The classical McNaughton Theorem of the previous subsection implies that free MV algebras can be represented as MV polynomials on $[0, 1]^n$.

However, these polynomials can also be represented as truncated infima of suprema of affine functions from $[0, 1]^n$ to $R$ with integer coefficients.

This idea can be extended to any MV-algebra $A$, so to relate truncated infima of suprema of affine functions from $A^n$ to $\Xi(A)$ (where $\Xi$ is the inverse Mundici functor, and MV-polynomial functions on $A$.

The advantage of this idea is that it works in an arbitrary MV-algebra, where we do not have the notion of continuity as we have in $[0, 1]$.

Let $A$ be an MV algebra with associated $\ell u$-group $(G, u)$. A $(G, u)$-*affine term* (with integer coefficients) from $A^n$ to $G$ is a term (in the language of groups) of the form

$f(x_1, \ldots, x_n) = g_0 + m_1 x_1 + \ldots + m_n x_n$, where $g_0 \in G$ and $m_1, \ldots, m_n \in Z$.

Note that we identify a variable $x_i$ with the corresponding projection.

Let $(G, u)$ be an $\ell u$-group associated to an $MV$ algebra $A$. For an element $g \in G$, we define the truncating function $\rho(g) = (g \vee 0) \wedge u$.

This defines a function $\rho : G \to A$.

A $(G,u)$-*term* is a term (in the language of $\ell$-groups) of the form $\bigvee_i \bigwedge_j f_{ij}(x)$, where $f_{ij}$ is affine, that is, a finite infimum of finite suprema of affine terms.

A *truncated* $(G,u)$ *term* is an expression of the form $\rho \circ t$, where $t$ is a $(G,u)$-term.

A truncated $(G,u)$-function is one defined by a truncated $(G,u)$-term.

We let $TF_n(G, u)$ be the set of all truncated $(G, u)$ functions in $n$ variables.

We note that the set $TF_n(G, u)$ of truncated $(G, u)$ functions is an MV-algebra.

In fact, we can define $t \oplus u = \rho \circ (t + u)$ and $\neg t = u - t$.

Since the inverse Mundici functor $\Xi$ gives a bijection between MV algebras and $\ell u$-groups, we can write without ambiguity $TF_n(A)$ for $TF_n(\Xi(A))$.

This notation will be useful in stating the next theorem, which clarifies:

the relation between truncated $(G, u)$-terms and MV-polynomials:

### Theorem

*Let $A$ be an MV algebra, with associated $\ell u$ group $(G, u)$. Then MV polynomials and truncated $(G, u)$-terms define the same functions from $A^n$ to $A$. That is,*

*the MV algebras $TF_n(A)$ and $\mathbf{PF}_n(A)$ coincide.*

**Polyhedra** and **McNaughton functions** for **MV − chains**:

We can exploit McNaughton Theorem to give the following characterization of zerosets of polynomials in MV chains.

Given an MV algebra $A$, an *affine function* on $A$ is a function of the form $\Sigma_j m_j x_j + r$, where $m_j$ are integers and $r \in \Xi(A)$.

### Proposition

*Let $A$ be an MV chain. The* **zerosets** *of a polynomial* $p(x_1, \ldots, x_n, a_1, \ldots, a_m) \in A[x_1, \ldots, x_n]$ **coincide** *with* **finite unions** *of* **polyhedra** *of the form*

$$\{x | a(x) \geq 0\},$$

*where $a(x)$ is an affine function on $A$.*

With the same kind of argument one can prove the following analogue of McNaughton Theorem itself for MV-chains.

Call McNaughton function over $A$ a function $f : A^n \to A$ for which there is a covering of $A^n$ by finitely many polyhedra $P_1, \ldots, P_k$ of the form $\{x | a(x) \geq 0\}$, such that $f$ on each polyhedron is affine.

### Proposition

*Let $A$ be an MV chain. Let $p \in A[x_1, \ldots, x_n]$. Then $p$ defines a McNaughton function from $A^n$ to $A$.*

*Conversely,*

*every McNaughton function from $A^n$ to $A$ is definable by a polynomial.*

**Algebraic Sets**:

In this section we focus on Diophantine algebraic geometry: that is, we take the same algebra $A$ both to define constants in polynomials and to evaluate polynomials.

Definition

Let $A$ be an MV-algebra. Let $S \subseteq A[x_1, \ldots, x_n]$, $S \neq \emptyset$.

Consider the set
$\{(a_1, \ldots, a_n) \in A^n \mid p(a_1, \ldots . a_n) = 0, \ \forall p(x_1, \ \ldots, \ x_n) \in S\}$.
Denote this set by $V(S)$, called the **algebraic set** determined by $S$.

Clearly if we let $I = id(S)$, the ideal of $A[x_1, \dots, x_n]$ generated by $S$, then $V(I) = V(S)$.

Thus

*algebraic sets are determined by ideals.*

### Definition

Call an ideal $J \subseteq A[x_1, \ldots, x_n]$ *singular* if $V(J) = \emptyset$. Otherwise call $J$ *non- singular*.

## Proposition

*Suppose we have a non-empty* $X \subseteq A^n$*. Then let* $I(X) = \{p \in A[x_1, \ldots, x_n] \mid p(\bar{y}) = 0, \ \forall \bar{y} \in X\}$ *where* $\bar{y} = (y_1, \ldots, y_n), \ y_i \in A$*.*

*Then* $I(X)$ *is an ideal of* $A[x_1, \ldots, x_n]$*.*

**Point ideals and point radicals**:

Call an ideal $J \subseteq A[\bar{x}]$ a **point ideal** if for some $\bar{a} = (a_1, \ldots, a_n) \in A^n$ we have $J = I(\bar{a})$.

We consider the fixpoints of the adjunction $(I, V)$:
For an ideal $I \subseteq A[\bar{x}]$ let ${}_{pt}\sqrt{I} = \bigcap\{I(\bar{a}) \mid I \subseteq I(\bar{a})\}$.

We call ${}_{pt}\sqrt{I}$ the **point radical** of $I$.

Note it is an ideal as well.

We have the following Nullstellensatz theorem:

### Theorem

*The ideals $J$ such that $I(V(J)) = J$ are exactly the point-radical ideals.*

### Proposition

*There is a one-one correspondence between* **point − radicals** *and* **algebraic sets**.

*Some properties of $V(I)$:*

In partial analogy with regular algebraic geometry over fields, the following is true:

**Proposition**

*Let $A$ be an MV algebra. Let $I, J, J_k$ be ideals of $A[x_1, \ldots, x_n]$. Then,*

1) $I \subseteq J$ *implies* $V(J) \subseteq V(I)$;
2) $V(0) = A^n, \; V(A[x_1, \ldots, \; x_n]) = \emptyset$;
3) *if $A$ is an MV chain then* $V(I \cap J) = V(I) \cup V(J)$;
4) $V(\Sigma_k J_k) = \bigcap_k V(J_k)$.

**Coordinate algebras**:

Here again we are in Diophantine geometry.

Definition

Let $Z \subseteq A^n$ be a non-empty algebraic set.

By the *co-ordinate MV-algebra* of $Z$ we mean the MV-algebra $A[\bar{x}]/I(Z)$.

Definition

Let $Z_1 \subseteq A^n,\ Z_2 \subseteq A^m$ be algebraic sets. A mapping $\varphi : Z_1 \to Z_2$ is called a **polynomial map**

iff

there are polynomials $p_1, \dots,\ p_m \in A[x_1, \dots,\ x_n]$ such that $\varphi(a_1, \dots,\ a_n) = (p_1(a_1, \dots,\ a_n), \dots,\ p_m(a_1, \dots,\ a_n))$ for every $(a_1, \dots, a_n) \in Z_1$.

Let $Z(A)$ be the collection of all algebraic subsets of $A^n$. Then with polynomial maps as morphisms, $Z(A)$ becomes a category.

We have the following duality:

Theorem

*The categories of all* **algebraic sets** *with polynomial maps as morphisms is dually isomorphic to the category of* **Coordinate MV − algebras**.

Let $Z = V(I)$, with $Z \neq \emptyset$;

let $F(Z,\ A)$ be the MV algebra of polynomial maps from $Z$ to $A$. Then:

**Proposition**

*$A[\bar{x}]/I$ is isomorphic to $F(Z,\ A)$.*

### Corollary

*Let $A$ be an MV algebra. Then $F(Z, A)$ is a quotient of $TF_n(A)$.*

**Logic of polynomials**:

The completeness theorem of Łukasiewicz infinite valued logic can be phrased in several ways. One way is this, for [0, 1] valued logic, if $\sigma$ is a wff in the variables $v_1, \dots, v_n$, and if the value of $\sigma$ for all values of the $v_i$ is always 1, then in the Lindenbaum algebra $[\sigma] = 1$, where $[\sigma]$ is the class of $\sigma$.

Now $[\sigma]$ can be interpretated as a function $[\sigma] : [0,\ 1]^n \rightarrow [0,\ 1]$ by $[\sigma](r_1, \ldots,\ r_n)$ equals the value of $\sigma$ with the assignment $v_i = r_i$. Note that this value is independent of the representative chosen form $[\sigma]$.

With this interpretation the completeness theorem can be phrased as: if the *function* $[\sigma]$ equals 1 on $[0,\ 1]^n$, then $[\sigma] = 1$ in the Lindenbaum algebra.

We can apply this idea to our context and we get what we call polynomial completeness.

We introduce the following notion:

### Definition

An MV algebra $A$ is **polynomially complete** if for every $n$, the only polynomial in $n$ variables inducing the zero function on $A^n$ is the zero polynomial.

Note that $A$ is polynomially complete if and only if ${}_{pt}\sqrt{0} = 0$ in $A[x_1, \ldots, x_n]$ for every $n$.

For instance we have:

### Proposition

*Every divisible MV chain is polynomially complete.*

For polynomially complete MV algebras, our generalized McNaughton Theorem becomes particularly interesting, because we can speak freely about polynomials rather than polynomial functions.

A characterization of polynomially complete MV chains:

We do not have a complete characterization of polynomially complete MV algebras, however in this paper we give one for MV chains.

### Theorem

*Let $C$ be an MV chain. The following are equivalent:*

1. *$C$ is polynomially complete;*
2. *every polynomial $p \in C[x_1, \dots, x_n]$ which induces the zero function on $C$ induces the zero function on $DH(C)$, where $DH(C)$ is the divisible hull of $C$.*

### Corollary

- *Every MV chain can be embedded in a polynomially complete MV chain.*
- *Every simple infinite MV chain is polynomially complete.*
- *No discrete MV chain $A$ is polynomially complete.*
- *No MV chain $A$ of finite rank is polynomially complete.*

Now it is natural to conjecture that the theorem extends to MV algebras:

Any MV algebra $A$ is polynomially complete

if and only if

every polynomial $p \in A[x_1, \ldots, x_n]$ which induces zero on $A^n$ induces zero also on $DH(A)^n$, where $DH(A)$ is the divisible hull of $A$.

Łukasiewicz logic with constants:

Like classical algebraic geometry, MV algebraic geometry can be studied by three different viewpoints: geometric (the algebraic sets), algebraic (coordinate algebras) and logical (theories and models).

While the first two approaches are studied in the previous sections of this paper, we are left with giving the basics of logic for Diophantine MV algebraic geometry.

We must define Łukasiewicz logic with constants in a fixed MV algebra $A$, which, according to the Diophantine approach, will be both the MV algebra where the constants of polynomials are taken and the MV algebra where polynomials are evaluated.

In order to begin the study of Łukasiewicz logic with constants in a fixed MV algebra $A$, denoted by $L_{\infty(A)}$, by adding constants denoting elements of $A$.

Like any other logic we must specify the syntax and semantics of $L_{\infty(A)}$. First, formulas are defined inductively as follows:

- variables $X_1, X_2, \ldots$ are formulas;
- constants $c_a$ for every $a \in A$ are formulas;
- if $\alpha$ is a formula, then $\neg\alpha$ is a formula;
- if $\alpha, \beta$ is a formula, then $\alpha \to \beta$ is a formula.

The semantics of $L_{\infty(A)}$ is given in terms of valuation functions $v$ from variables to elements of $A$. The value of a formula $\alpha$ in a valuation $v$ is an element $v(\varphi)$ of $A$ defined by:

- $v(X_i)$ when $X_i$ is a variable;
- $a$ when the formula is the constant $c_a$;
- $v(\neg\alpha) = \neg v(\alpha)$;
- $v(\alpha \to \beta) = v(\alpha) \to v(\beta)$.

Now the notions of satisfaction, model, tautology, semantic consequence are like Łukasiewicz Logic. In particular, a model of a formula $\alpha$ is a valuation $v$ such that $v(\alpha) = 1$. A formula $\alpha$ is a tautology if $v(\alpha) = 1$ for every valuation $v$.
A formula $\alpha$ is a semantic consequence of a set of formulas $\Theta$ if every model of $\Theta$ is also a model of $\alpha$.

In $L_{\infty(A)}$ we give also a deductive system, extending the one of (CDM), section 4.3 with axioms for constants. The axioms are:

- $\alpha \to (\beta \to \alpha)$;
- $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$;
- $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$;
- $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$;
- $c_{a^* \oplus b} \to (c_a \to c_b)$;
- $(c_a \to c_b) \to c_{a^* \oplus b}$;
- $c_{a^*} \to \neg c_a$;
- $\neg c_a \to c_{a^*}$.

The only rule is Modus ponens, defined as usual: from $\alpha$ and $\alpha \to \beta$ derive $\beta$.

The notions of provable formula, proof, possibly with hypotheses, and theory are standard. The same holds for Lindenbaum MV algebra. We denote by $Lind(A)$ the Lindenbaum algebra of $L_{\infty(A)}$: that is, the set of all formulas of $L_{\infty(A)}$ modulo mutual provability. However, $Lind(A)$ is simply the polynomial MV algebra in countably many variables:

**Theorem**

*For every MV algebra $A$, the MV algebras $Lind(A)$ and $A[x_1, x_2, \ldots]$ are isomorphic.*

We will say that a logic is complete if tautologies coincide with provable formulas (by logic here we mean any set of strings equipped with a deductive system and a set of valuation functions taking values in one or more MV algebras).

Clearly, for every $A$, every provable formula of $L_{\infty(A)}$ is a tautology. The converse implication does not hold in general, but we have a characterization in terms of polynomial completeness:

**Theorem**

*For every MV algebra $A$, the logic $L_{\infty(A)}$ is complete if and only if $A$ is polynomially complete.*

We can summarize the main results as follows:

- We identify polynomial functions over any MV algebra with a kind of truncated functions;
- we give a form of Nullstellensatz for $A[x_1, \dots, x_n]$;
- we give a universal algebraic duality between algebraic sets and their coordinate algebras;
- we introduce the definition of polynomially complete MV algebra (i.e. one where polynomials and polynomial functions coincide) and we give a characterization of polynomially complete MV chains;
- we give a completeness theorem for Łukasiewicz logic with constants.

The results obtained so far suggest that a study of non-Diophantine algebraic geometry for MV algebras deserves to be pursued.