

Decision Making under Interval Uncertainty

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1. Decision Making: General Need and Traditional Approach

- To make a decision, we must:
 - find out the user's preference, and
 - help the user select an alternative which is the best
 - according to these preferences.
- Traditional approach is based on an assumption that for each two alternatives A' and A'' , a user can tell:
 - whether the first alternative is better for him/her; we will denote this by $A'' < A'$;
 - or the second alternative is better; we will denote this by $A' < A''$;
 - or the two given alternatives are of equal value to the user; we will denote this by $A' = A''$.

2. The Notion of Utility

- Under the above assumption, we can form a natural numerical scale for describing preferences.
- Let us select a very bad alternative A_0 and a very good alternative A_1 .
- Then, most other alternatives are better than A_0 but worse than A_1 .
- For every prob. $p \in [0, 1]$, we can form a lottery $L(p)$ in which we get A_1 w/prob. p and A_0 w/prob. $1 - p$.
- When $p = 0$, this lottery simply coincides with the alternative A_0 : $L(0) = A_0$.
- The larger the probability p of the positive outcome increases, the better the result:

$$p' < p'' \text{ implies } L(p') < L(p'').$$

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3. The Notion of Utility (cont-d)

- Finally, for $p = 1$, the lottery coincides with the alternative A_1 : $L(1) = A_1$.
- Thus, we have a continuous scale of alternatives $L(p)$ that monotonically goes from $L(0) = A_0$ to $L(1) = A_1$.
- Due to monotonicity, when p increases, we first have $L(p) < A$, then we have $L(p) > A$.
- The threshold value is called the *utility* of the alternative A :

$$u(A) \stackrel{\text{def}}{=} \sup\{p : L(p) < A\} = \inf\{p : L(p) > A\}.$$

- Then, for every $\varepsilon > 0$, we have

$$L(u(A) - \varepsilon) < A < L(u(A) + \varepsilon).$$

- We will describe such (almost) equivalence by \equiv , i.e., we will write that $A \equiv L(u(A))$.

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4. Fast Iterative Process for Determining $u(A)$

- *Initially:* we know the values $\underline{u} = 0$ and $\bar{u} = 1$ such that $A \equiv L(u(A))$ for some $u(A) \in [\underline{u}, \bar{u}]$.
- *What we do:* we compute the midpoint u_{mid} of the interval $[\underline{u}, \bar{u}]$ and compare A with $L(u_{\text{mid}})$.
- *Possibilities:* $A \leq L(u_{\text{mid}})$ and $L(u_{\text{mid}}) \leq A$.
- *Case 1:* if $A \leq L(u_{\text{mid}})$, then $u(A) \leq u_{\text{mid}}$, so
$$u \in [\underline{u}, u_{\text{mid}}].$$
- *Case 2:* if $L(u_{\text{mid}}) \leq A$, then $u_{\text{mid}} \leq u(A)$, so
$$u \in [u_{\text{mid}}, \bar{u}].$$
- After each iteration, we decrease the width of the interval $[\underline{u}, \bar{u}]$ by half.
- After k iterations, we get an interval of width 2^{-k} which contains $u(A)$ – i.e., we get $u(A)$ w/accuracy 2^{-k} .

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5. How to Make a Decision Based on Utility Values

- Suppose that we have found the utilities $u(A')$, $u(A'')$, \dots , of the alternatives A' , A'' , \dots
- Which of these alternatives should we choose?
- By definition of utility, we have:
 - $A \equiv L(u(A))$ for every alternative A , and
 - $L(p') < L(p'')$ if and only if $p' < p''$.
- We can thus conclude that A' is preferable to A'' if and only if $u(A') > u(A'')$.
- In other words, we should always select an alternative with the largest possible value of utility.
- Interval techniques can help in finding the optimizing decision.

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6. How to Estimate Utility of an Action

- For each action, we usually know possible outcomes S_1, \dots, S_n .
- We can often estimate the prob. p_1, \dots, p_n of these outcomes.
- By definition of utility, each situation S_i is equiv. to a lottery $L(u(S_i))$ in which we get:
 - A_1 with probability $u(S_i)$ and
 - A_0 with the remaining probability $1 - u(S_i)$.
- Thus, the action is equivalent to a complex lottery in which:
 - first, we select one of the situations S_i with probability p_i : $P(S_i) = p_i$;
 - then, depending on S_i , we get A_1 with probability $P(A_1 | S_i) = u(S_i)$ and A_0 w/probability $1 - u(S_i)$.

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7. How to Estimate Utility of an Action (cont-d)

- *Reminder:*

- first, we select one of the situations S_i with probability p_i : $P(S_i) = p_i$;
- then, depending on S_i , we get A_1 with probability $P(A_1 | S_i) = u(S_i)$ and A_0 w/probability $1 - u(S_i)$.

- The prob. of getting A_1 in this complex lottery is:

$$P(A_1) = \sum_{i=1}^n P(A_1 | S_i) \cdot P(S_i) = \sum_{i=1}^n u(S_i) \cdot p_i.$$

- In the complex lottery, we get:

- A_1 with prob. $u = \sum_{i=1}^n p_i \cdot u(S_i)$, and
- A_0 w/prob. $1 - u$.

- So, we should select the action with the largest value of expected utility $u = \sum p_i \cdot u(S_i)$.

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8. Non-Uniqueness of Utility

- The above definition of utility u depends on A_0, A_1 .
- What if we use different alternatives A'_0 and A'_1 ?
- Every A is equivalent to a lottery $L(u(A))$ in which we get A_1 w/prob. $u(A)$ and A_0 w/prob. $1 - u(A)$.
- For simplicity, let us assume that $A'_0 < A_0 < A_1 < A'_1$.
- Then, $A_0 \equiv L'(u'(A_0))$ and $A_1 \equiv L'(u'(A_1))$.
- So, A is equivalent to a complex lottery in which:
 - 1) we select A_1 w/prob. $u(A)$ and A_0 w/prob. $1 - u(A)$;
 - 2) depending on A_i , we get A'_1 w/prob. $u'(A_i)$ and A'_0 w/prob. $1 - u'(A_i)$.
- In this complex lottery, we get A'_1 with probability $u'(A) = u(A) \cdot (u'(A_1) - u'(A_0)) + u'(A_0)$.
- So, in general, utility is defined modulo an (increasing) linear transformation $u' = a \cdot u + b$, with $a > 0$.

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9. Subjective Probabilities

- In practice, we often do not know the probabilities p_i of different outcomes.
- For each event E , a natural way to estimate its subjective probability is to fix a prize (e.g., \$1) and compare:
 - the lottery ℓ_E in which we get the fixed prize if the event E occurs and 0 if it does not occur, with
 - a lottery $\ell(p)$ in which we get the same amount with probability p .
- Here, similarly to the utility case, we get a value $ps(E)$ for which, for every $\varepsilon > 0$:

$$\ell(ps(E) - \varepsilon) < \ell_E < \ell(ps(E) + \varepsilon).$$

- Then, the utility of an action with possible outcomes S_1, \dots, S_n is equal to $u = \sum_{i=1}^n ps(E_i) \cdot u(S_i)$.

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10. Beyond Traditional Decision Making: Towards a More Realistic Description

- Previously, we assumed that a user can always decide which of the two alternatives A' and A'' is better:
 - either $A' < A''$,
 - or $A'' < A'$,
 - or $A' \equiv A''$.
- In practice, a user is sometimes unable to meaningfully decide between the two alternatives; denoted $A' \parallel A''$.
- In mathematical terms, this means that the preference relation:
 - is no longer a *total* (linear) order,
 - it can be a *partial* order.

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11. From Utility to Interval-Valued Utility

- Similarly to the traditional decision making approach:
 - we select two alternatives $A_0 < A_1$ and
 - we compare each alternative A which is better than A_0 and worse than A_1 with lotteries $L(p)$.

- Since preference is a *partial* order, in general:

$$\underline{u}(A) \stackrel{\text{def}}{=} \sup\{p : L(p) < A\} < \bar{u}(A) \stackrel{\text{def}}{=} \inf\{p : L(p) > A\}.$$

- For each alternative A , instead of a single value $u(A)$ of the utility, we now have an *interval* $[\underline{u}(A), \bar{u}(A)]$ s.t.:
 - if $p < \underline{u}(A)$, then $L(p) < A$;
 - if $p > \bar{u}(A)$, then $A < L(p)$; and
 - if $\underline{u}(A) < p < \bar{u}(A)$, then $A \parallel L(p)$.
- We will call this interval the *utility* of the alternative A .

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12. Interval-Valued Utilities and Interval-Valued Subjective Probabilities

- To feasibly elicit the values $\underline{u}(A)$ and $\bar{u}(A)$, we:
 - 1) starting w/ $[\underline{u}, \bar{u}] = [0, 1]$, bisect an interval s.t. $L(\underline{u}) < A < L(\bar{u})$ until we find u_0 s.t. $A \parallel L(u_0)$;
 - 2) by bisecting an interval $[\underline{u}, u_0]$ for which $L(\underline{u}) < A \parallel L(u_0)$, we find $\underline{u}(A)$;
 - 3) by bisecting an interval $[u_0, \bar{u}]$ for which $L(u_0) \parallel A < L(\bar{u})$, we find $\bar{u}(A)$.
- Similarly, when we estimate the probability of an event E :
 - we no longer get a single value $ps(E)$;
 - we get an *interval* $[\underline{ps}(E), \bar{ps}(E)]$ of possible values of probability.
- By using bisection, we can feasibly elicit the values $\underline{ps}(E)$ and $\bar{ps}(E)$.

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13. Decision Making Under Interval Uncertainty

- *Situation*: for each possible decision d , we know the interval $[\underline{u}(d), \bar{u}(d)]$ of possible values of utility.
- *Questions*: which decision shall we select?
- *Natural idea*: select all decisions d_0 that *may* be optimal, i.e., which are optimal for some function

$$u(d) \in [\underline{u}(d), \bar{u}(d)].$$

- *Problem*: checking all possible functions is not feasible.
- *Solution*: the above condition is equivalent to an easier-to-check one:

$$\bar{u}(d_0) \geq \max_d \underline{u}(d).$$

- *Interval computations* can help in describing the range of all such d_0 .
- *Remaining problem*: in practice, we would like to select *one* decision; which one should be select?

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14. Need for Definite Decision Making

- *At first glance:* if $A' \parallel A''$, it does not matter whether we recommend alternative A' or alternative A'' .
- Let us show that this is *not* a good recommendation.
- E.g., let A be an alternative about which we know nothing, i.e., $[\underline{u}(A), \bar{u}(A)] = [0, 1]$.
- In this case, A is indistinguishable both from a “good” lottery $L(0.999)$ and a “bad” lottery $L(0.001)$.
- Suppose that we recommend, to the user, that A is equivalent both to $L(0.999)$ and to $L(0.001)$.
- Then this user will feel comfortable:
 - first, exchanging $L(0.999)$ with A , and
 - then, exchanging A with $L(0.001)$.
- So, following our recommendations, the user switches from a very good alternative to a very bad one.

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15. Need for Definite Decision Making (cont-d)

- The above argument does not depend on the fact that we assumed complete ignorance about A :
 - every time we recommend that the alternative A is “equivalent” both to $L(p)$ and to $L(p')$ ($p < p'$),
 - we make the user vulnerable to a similar switch from a better alternative $L(p')$ to a worse one $L(p)$.
- Thus, there should be only a single value p for which A can be reasonably exchanged with $L(p)$.
- In precise terms:
 - we start with the utility interval $[\underline{u}(A), \bar{u}(A)]$, and
 - we need to select a single $u(A)$ for which it is reasonable to exchange A with a lottery $L(u)$.
- How can we find this value $u(A)$?

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16. Decisions under Interval Uncertainty: Hurwicz Optimism-Pessimism Criterion

- *Reminder:* we need to assign, to each interval $[\underline{u}, \bar{u}]$, a utility value $u(\underline{u}, \bar{u}) \in [\underline{u}, \bar{u}]$.
- *History:* this problem was first handled in 1951, by the future Nobelist Leonid Hurwicz.
- *Notation:* let us denote $\alpha_H \stackrel{\text{def}}{=} u(0, 1)$.
- *Reminder:* utility is determined modulo a linear transformation $u' = a \cdot u + b$.
- *Reasonable to require:* the equivalent utility does not change with re-scaling: for $a > 0$ and b ,

$$u(a \cdot u^- + b, a \cdot u^+ + b) = a \cdot u(u^-, u^+) + b.$$

- For $u^- = 0$, $u^+ = 1$, $a = \bar{u} - \underline{u}$, and $b = \underline{u}$, we get

$$u(\underline{u}, \bar{u}) = \alpha_H \cdot (\bar{u} - \underline{u}) + \underline{u} = \alpha_H \cdot \bar{u} + (1 - \alpha_H) \cdot \underline{u}.$$

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17. Hurwicz Optimism-Pessimism Criterion (cont)

- The expression $\alpha_H \cdot \bar{u} + (1 - \alpha_H) \cdot \underline{u}$ is called *optimism-pessimism criterion*, because:
 - when $\alpha_H = 1$, we make a decision based on the most optimistic possible values $u = \bar{u}$;
 - when $\alpha_H = 0$, we make a decision based on the most pessimistic possible values $u = \underline{u}$;
 - for intermediate values $\alpha_H \in (0, 1)$, we take a weighted average of the optimistic and pessimistic values.
- According to this criterion:
 - if we have several alternatives A', \dots , with interval-valued utilities $[\underline{u}(A'), \bar{u}(A')]$, \dots ,
 - we recommend an alternative A that maximizes

$$\alpha_H \cdot \bar{u}(A) + (1 - \alpha_H) \cdot \underline{u}(A).$$

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18. Which Value α_H Should We Choose? An Argument in Favor of $\alpha_H = 0.5$

- Let us take an event E about which we know nothing.
- For a lottery L^+ in which we get A_1 if E and A_0 otherwise, the utility interval is $[0, 1]$.
- Thus, the equiv. utility of L^+ is $\alpha_H \cdot 1 + (1 - \alpha_H) \cdot 0 = \alpha_H$.
- For a lottery L^- in which we get A_0 if E and A_1 otherwise, the utility is $[0, 1]$, so equiv. utility is also α_H .
- For a complex lottery L in which we select either L^+ or L^- with probability 0.5, the equiv. utility is still α_H .
- On the other hand, in L , we get A_1 with probability 0.5 and A_0 with probability 0.5.
- Thus, $L = L(0.5)$ and hence, $u(L) = 0.5$.
- So, we conclude that $\alpha_H = 0.5$.

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19. Which Action Should We Choose?

- Suppose that an action has n possible outcomes S_1, \dots, S_n , with utilities $[\underline{u}(S_i), \bar{u}(S_i)]$, and probabilities $[\underline{p}_i, \bar{p}_i]$.
- We know that each alternative is equivalent to a simple lottery with utility $u_i = \alpha_H \cdot \bar{u}(S_i) + (1 - \alpha_H) \cdot \underline{u}(S_i)$.
- We know that for each i , the i -th event is equivalent to $p_i = \alpha_H \cdot \bar{p}_i + (1 - \alpha_H) \cdot \underline{p}_i$.
- Thus, this action is equivalent to a situation in which we get utility u_i with probability p_i .
- The utility of such a situation is equal to $\sum_{i=1}^n p_i \cdot u_i$.
- Thus, the equivalent utility of the original action is equivalent to

$$\sum_{i=1}^n \left(\alpha_H \cdot \bar{p}_i + (1 - \alpha_H) \cdot \underline{p}_i \right) \cdot \left(\alpha_H \cdot \bar{u}(S_i) + (1 - \alpha_H) \cdot \underline{u}(S_i) \right).$$

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20. Observation: the Resulting Decision Depends on the Level of Detail

- Let us consider a situation in which, with some prob. p , we gain a utility u , else we get 0.
- The expected utility is $p \cdot u + (1 - p) \cdot 0 = p \cdot u$.
- Suppose that we only know the intervals $[\underline{u}, \bar{u}]$ and $[\underline{p}, \bar{p}]$.

- The equivalent utility u_k (k for *know*) is

$$u_k = (\alpha_H \cdot \bar{p} + (1 - \alpha_H) \cdot \underline{p}) \cdot (\alpha_H \cdot \bar{u} + (1 - \alpha_H) \cdot \underline{u}).$$

- If we only know that utility is from $[\underline{p} \cdot \underline{u}, \bar{p} \cdot \bar{u}]$, then:

$$u_d = \alpha_H \cdot \bar{p} \cdot \bar{u} + (1 - \alpha_H) \cdot \underline{p} \cdot \underline{u} \quad (d \text{ for } \textit{don't know}).$$

- Here, additional knowledge decreases utility:

$$u_d - u_k = \alpha_H \cdot (1 - \alpha_H) \cdot (\bar{p} - \underline{p}) \cdot (\bar{u} - \underline{u}) > 0.$$

- (This is maybe what the Book of Ecclesiastes meant by “For with much wisdom comes much sorrow”?)

21. Beyond Interval Uncertainty: Partial Info about Probabilities

- *Frequent situation*:
 - in addition to \mathbf{x}_i ,
 - we may also have *partial* information about the probabilities of different values $x_i \in \mathbf{x}_i$.
- An *exact* probability distribution can be described, e.g., by its cumulative distribution function

$$F_i(z) = \text{Prob}(x_i \leq z).$$

- A *partial* information means that instead of a single cdf, we have a *class* \mathcal{F} of possible cdfs.
- *p-box* (Scott Ferson):
 - for every z , we know an interval $\mathbf{F}(z) = [\underline{F}(z), \overline{F}(z)]$;
 - we consider all possible distributions for which, for all z , we have $F(z) \in \mathbf{F}(z)$.

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22. Describing Partial Info about Probabilities: Decision Making Viewpoint

- *Problem:* there are many ways to represent a probability distribution.
- *Idea:* look for an objective.
- *Objective:* make decisions $E_x[u(x, a)] \rightarrow \max_a$.
- *Case 1:* smooth $u(x)$.
- *Analysis:* we have $u(x) = u(x_0) + (x - x_0) \cdot u'(x_0) + \dots$
- *Conclusion:* we must know moments to estimate $E[u]$.
- *Case of uncertainty:* interval bounds on moments.
- *Case 2:* threshold-type $u(x)$ (e.g., regulations).
- *Conclusion:* we need cdf $F(x) = \text{Prob}(\xi \leq x)$.
- *Case of uncertainty:* p-box $[\underline{F}(x), \overline{F}(x)]$.

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23. What if Intervals are Difficult to Elicit

- *Problem:* in some situations, it is difficult to elicit even interval-valued utilities.
- *Case study:* selecting a location for a meteorological tower.
- *What we can use for decision making:* in many such situations, there are reasonable symmetries.
- *Good news:* in such cases, we can often use symmetries to select an optimal decision.
- *We show:* how this works on the case study example.

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24. Case Study

- *Objective*: select the best location of a sophisticated multi-sensor meteorological tower.
- *Constraints*: we have several criteria to satisfy.
- *Example*: the station should not be located too close to a road.
- *Motivation*: the gas flux generated by the cars do not influence our measurements of atmospheric fluxes.
- *Formalization*: the distance x_1 to the road should be larger than a threshold t_1 : $x_1 > t_1$, or $y_1 \stackrel{\text{def}}{=} x_1 - t_1 > 0$.
- *Example*: the inclination x_2 at the tower's location should be smaller than a threshold t_2 : $x_2 < t_2$.
- *Motivation*: otherwise, the flux determined by this inclination and not by atmospheric processes.

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25. General Case

- *In general*: we have several differences y_1, \dots, y_n all of which have to be non-negative.
- For each of the differences y_i , the larger its value, the better.
- Our problem is a typical setting for *multi-criteria optimization*.
- A most widely used approach to multi-criteria optimization is *weighted average*, where
 - we assign weights $w_1, \dots, w_n > 0$ to different criteria y_i and
 - select an alternative for which the weighted average

$$w_1 \cdot y_1 + \dots + w_n \cdot y_n$$

attains the largest possible value.

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26. Limitations of the Weighted Average Approach

- *In general:* the weighted average approach often leads to reasonable solutions of the multi-criteria problem.
- *In our problem:* we have an additional requirement – that all the values y_i must be positive. So:
 - when selecting an alternative with the largest possible value of the weighted average,
 - we must only compare solutions with $y_i > 0$.
- *We will show:* under the requirement $y_i > 0$, the weighted average approach is not fully satisfactory.
- *Conclusion:* we need to find a more adequate solution.

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27. Limitations of the Weighted Average Approach: Details

- The values y_i come from measurements, and measurements are never absolutely accurate.
- The results \tilde{y}_i of the measurements are not exactly equal to the actual (unknown) values y_i .
- *If:* for some alternative $y = (y_1, \dots, y_n)$
 - we measure the values y_i with higher and higher accuracy and,
 - based on the measurement results \tilde{y}_i , we conclude that y is better than some other alternative y' .
- *Then:* we expect that the actual alternative y is indeed better than y' (or at least of the same quality).
- Otherwise, we will not be able to make any meaningful conclusions based on real-life measurements.

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28. The Above Natural Requirement Is Not Always Satisfied for Weighted Average

- *Simplest case:* two criteria y_1 and y_2 , w/weights $w_i > 0$.
- If $y_1, y_2, y'_1, y'_2 > 0$, and $w_1 \cdot y_1 + w_2 \cdot y_2 > w_1 \cdot y'_1 + w_2 \cdot y'_2$, then $y = (y_1, y_2) \succ y' = (y'_1, y'_2)$.

- If $y_1 > 0, y_2 > 0$, and at least one of the values y'_1 and y'_2 is non-positive, then $y = (y_1, y_2) \succ y' = (y'_1, y'_2)$.

- Let us consider, for every $\varepsilon > 0$, the tuple $y(\varepsilon) \stackrel{\text{def}}{=} (\varepsilon, 1 + w_1/w_2)$, and $y' = (1, 1)$.

- In this case, for every $\varepsilon > 0$, we have

$$w_1 \cdot y_1(\varepsilon) + w_2 \cdot y_2(\varepsilon) = w_1 \cdot \varepsilon + w_2 + w_2 \cdot \frac{w_1}{w_2} = w_1 \cdot (1 + \varepsilon) + w_2$$

and $w_1 \cdot y'_1 + w_2 \cdot y'_2 = w_1 + w_2$, hence $y(\varepsilon) \succ y'$.

- However, in the limit $\varepsilon \rightarrow 0$, we have $y(0) = \left(0, 1 + \frac{w_1}{w_2}\right)$, with $y(0)_1 = 0$ and thus, $y(0) \prec y'$.

29. Towards a Precise Description

- Each alternative is characterized by a tuple of n positive values $y = (y_1, \dots, y_n)$.
- Thus, the set of all alternatives is the set $(R^+)^n$ of all the tuples of positive numbers.
- For each two alternatives y and y' , we want to tell whether
 - y is better than y' (we will denote it by $y \succ y'$ or $y' \prec y$),
 - or y' is better than y ($y' \succ y$),
 - or y and y' are equally good ($y' \sim y$).
- *Natural requirement:* if y is better than y' and y' is better than y'' , then y is better than y'' .
- The relation \succ must be transitive.

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30. Towards a Precise Description (cont-d)

- *Reminder*: the relation \succ must be transitive.
- Similarly, the relation \sim must be transitive, symmetric, and reflexive ($y \sim y$), i.e., be an *equivalence relation*.
- *An alternative description*: a transitive pre-ordering relation $a \succeq b \Leftrightarrow (a \succ b \vee a \sim b)$ s.t. $a \succeq b \vee b \succeq a$.

- Then, $a \sim b \Leftrightarrow (a \succeq b) \& (b \succeq a)$, and

$$a \succ b \Leftrightarrow (a \succeq b) \& (b \not\succeq a).$$

- *Additional requirement*:
 - if each criterion is better,
 - then the alternative is better as well.
- *Formalization*: if $y_i > y'_i$ for all i , then $y \succ y'$.

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31. Scale Invariance: Motivation

- *Fact:* quantities y_i describe completely different physical notions, measured in completely different units.
- *Examples:* wind velocities measured in m/s, km/h, mi/h; elevations in m, km, ft.
- Each of these quantities can be described in many different units.
- A priori, we do not know which units match each other.
- Units used for measuring different quantities may not be exactly matched.
- It is reasonable to require that:
 - if we simply change the units in which we measure each of the corresponding n quantities,
 - the relations \succ and \sim between the alternatives $y = (y_1, \dots, y_n)$ and $y' = (y'_1, \dots, y'_n)$ do not change.

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32. Scale Invariance: Towards a Precise Description

- *Situation*: we replace:
 - a unit in which we measure a certain quantity q
 - by a new measuring unit which is $\lambda > 0$ times smaller.
- *Result*: the numerical values of this quantity increase by a factor of λ : $q \rightarrow \lambda \cdot q$.
- *Example*: 1 cm is $\lambda = 100$ times smaller than 1 m, so the length $q = 2$ becomes $\lambda \cdot q = 2 \cdot 100 = 200$ cm.
- Then, scale-invariance means that for all $y, y' \in (R^+)^n$ and for all $\lambda_i > 0$, we have
 - $y = (y_1, \dots, y_n) \succ y' = (y'_1, \dots, y'_n)$ implies $(\lambda_1 \cdot y_1, \dots, \lambda_n \cdot y_n) \succ (\lambda_1 \cdot y'_1, \dots, \lambda_n \cdot y'_n)$,
 - $y = (y_1, \dots, y_n) \sim y' = (y'_1, \dots, y'_n)$ implies $(\lambda_1 \cdot y_1, \dots, \lambda_n \cdot y_n) \sim (\lambda_1 \cdot y'_1, \dots, \lambda_n \cdot y'_n)$.

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33. Formal Description

- By a *total pre-ordering relation* on a set Y , we mean
 - a pair of a transitive relation \succ and an equivalence relation \sim for which,
 - for every $y, y' \in Y$, exactly one of the following relations hold: $y \succ y'$, $y' \succ y$, or $y \sim y'$.
- We say that a total pre-ordering is *non-trivial* if there exist y and y' for which $y \succ y'$.
- We say that a total pre-ordering relation on $(R^+)^n$ is:
 - *monotonic* if $y'_i > y_i$ for all i implies $y' \succ y$;
 - *continuous* if
 - * whenever we have a sequence $y^{(k)}$ of tuples for which $y^{(k)} \succeq y'$ for some tuple y' , and
 - * the sequence $y^{(k)}$ tends to a limit y ,
 - * then $y \succeq y'$.

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34. Main Result

Theorem. *Every non-trivial monotonic scale-inv. continuous total pre-ordering relation on $(\mathbb{R}^+)^n$ has the form:*

$$y' = (y'_1, \dots, y'_n) \succ y = (y_1, \dots, y_n) \Leftrightarrow \prod_{i=1}^n (y'_i)^{\alpha_i} > \prod_{i=1}^n y_i^{\alpha_i};$$

$$y' = (y'_1, \dots, y'_n) \sim y = (y_1, \dots, y_n) \Leftrightarrow \prod_{i=1}^n (y'_i)^{\alpha_i} = \prod_{i=1}^n y_i^{\alpha_i},$$

for some constants $\alpha_i > 0$.

Comment: Vice versa,

- for each set of values $\alpha_1 > 0, \dots, \alpha_n > 0$,
- the above formulas define a monotonic scale-invariant continuous pre-ordering relation on $(\mathbb{R}^+)^n$.

35. Practical Conclusion

- *Situation:*
 - we need to select an alternative;
 - each alternative is characterized by characteristics y_1, \dots, y_n .
- *Traditional approach:*
 - we assign the weights w_i to different characteristics;
 - we select the alternative with the largest value of
$$\sum_{i=1}^n w_i \cdot y_i.$$
- *New result:* it is better to select an alternative with the largest value of
$$\prod_{i=1}^n y_i^{w_i}.$$
- *Equivalent reformulation:* select an alternative with the largest value of
$$\sum_{i=1}^n w_i \cdot \ln(y_i).$$

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36. Multi-Agent Cooperative Decision Making

- *How to describe preferences:* for each participant P_i , we can determine the utility $u_{ij} \stackrel{\text{def}}{=} u_i(A_j)$ of all A_j .
- *Question:* how to transform these utilities into a reasonable group decision rule?
- *Solution:* was provided by another future Nobelist John Nash.
- *Nash's assumptions:*
 - symmetry,
 - independence from irrelevant alternatives, and
 - *scale invariance* – under replacing function $u_i(A)$ with an equivalent function $a \cdot u_i(A)$,

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37. Nash's Bargaining Solution (cont-d)

- *Nash's assumptions (reminder):*
 - symmetry,
 - independence from irrelevant alternatives, and
 - scale invariance.
- *Nash's result:*
 - the only group decision rule satisfying all these assumptions
 - is selecting an alternative A for which the product $\prod_{i=1}^n u_i(A)$ is the largest possible.
- *Comment.* the utility functions must be “scaled” s.t. the “status quo” situation $A^{(0)}$ has utility 0:

$$u_i(A) \rightarrow u'_i(A) \stackrel{\text{def}}{=} u_i(A) - u_i(A^{(0)}).$$

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38. Multi-Agent Decision Making under Interval Uncertainty

- *Reminder:* if we set utility of status quo to 0, then we select an alternative A that maximizes

$$u(A) = \prod_{i=1}^n u_i(A).$$

- *Case of interval uncertainty:* we only know intervals $[\underline{u}_i(A), \bar{u}_i(A)]$.
- *First idea:* find all A_0 for which $\bar{u}(A_0) \geq \max_A \underline{u}(A)$, where

$$[\underline{u}(A), \bar{u}(A)] \stackrel{\text{def}}{=} \prod_{i=1}^n [\underline{u}_i(A), \bar{u}_i(A)].$$

- *Second idea:* maximize $u^{\text{equiv}}(A) \stackrel{\text{def}}{=} \prod_{i=1}^n u_i^{\text{equiv}}(A)$.
- *Interesting aspect:* when we have a conflict situation (e.g., in security games).

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39. Beyond Optimization

- *Traditional interval computations:*

- we know the intervals X_1, \dots, X_n containing x_1, \dots, x_n ;
- we know that a quantity z depends on $x = (x_1, \dots, x_n)$:

$$z = f(x_1, \dots, x_n);$$

- we want to find the range Z of possible values of z :

$$Z = \left[\min_{x \in X} f(x), \max_{x \in X} f(x) \right].$$

- *Control situations:*

- the value $z = f(x, u)$ also depends on the control variables $u = (u_1, \dots, u_m)$;
- we want to find Z for which, for every $x_i \in X_i$, we can get $z \in Z$ by selecting appropriate $u_j \in U_j$:

$$\forall x \exists u (z = f(x, u) \in Z).$$

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40. Reformulation in Logical Terms – of Modal Intervals

- *Reminder:* we want $\forall x \in X \exists u \in U (f(x, u) \in Z)$.
- There is a logical difference between intervals X and U .
- The property $f(x, u) \in Z$ must hold
 - for all possible values $x_i \in X_i$, but
 - for some values $u_j \in U_j$.
- We can thus consider pairs of intervals and quantifiers (*modal intervals*):
 - each original interval X_i is a pair $\langle X_i, \forall \rangle$, while
 - controlled interval is a pair $\langle U_j, \exists \rangle$.
- We can treat the resulting interval Z as the range defined over modal intervals:

$$Z = f(\langle X_1, \forall \rangle, \dots, \langle X_n, \forall \rangle, \langle U_1, \exists \rangle, \dots, \langle U_m, \exists \rangle).$$

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41. Even Further Beyond Optimization

- In more complex situations, we need to go beyond control.
- For example, in the presence of an adversary, we want to make a decision x such that:
 - for every possible reaction y of an adversary,
 - we will be able to make a next decision x' (depending on y)
 - so that after every possible next decision y' of an adversary,
 - the resulting state $s(x, y, x', y')$ will be in the desired set:

$$\forall y \exists x' \forall y' (s(x, y, x', y') \in S).$$

- In this case, we arrive at general Shary's classes.

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42. Acknowledgments

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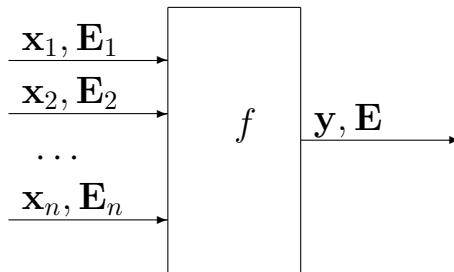
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43. Extension of Interval Arithmetic to Probabilistic Case: Successes

- *General solution:* parse to elementary operations $+$, $-$, \cdot , $1/x$, \max , \min .
- Explicit formulas for arithmetic operations are known:
 - for intervals,
 - for p-boxes $\mathbf{F}(x) = [\underline{F}(x), \overline{F}(x)]$,
 - for intervals + 1st moments $E_i \stackrel{\text{def}}{=} E[x_i]$:



44. Extension of Interval Arithmetic to Probabilistic Case: Successes (cont-d)

- *Easy cases:* +, −, product of independent x_i .
- *Example of a non-trivial case:* multiplication $y = x_1 \cdot x_2$, when we have no info about correlation.
- *Solution for this case:* for $p_i \stackrel{\text{def}}{=} (E_i - \underline{x}_i)/(\bar{x}_i - \underline{x}_i)$, we get:

- $\underline{E} = \max(p_1 + p_2 - 1, 0) \cdot \bar{x}_1 \cdot \bar{x}_2 + \min(p_1, 1 - p_2) \cdot \bar{x}_1 \cdot \underline{x}_2 + \min(1 - p_1, p_2) \cdot \underline{x}_1 \cdot \bar{x}_2 + \max(1 - p_1 - p_2, 0) \cdot \underline{x}_1 \cdot \underline{x}_2$;
- $\bar{E} = \min(p_1, p_2) \cdot \bar{x}_1 \cdot \bar{x}_2 + \max(p_1 - p_2, 0) \cdot \bar{x}_1 \cdot \underline{x}_2 + \max(p_2 - p_1, 0) \cdot \underline{x}_1 \cdot \bar{x}_2 + \min(1 - p_1, 1 - p_2) \cdot \underline{x}_1 \cdot \underline{x}_2$.

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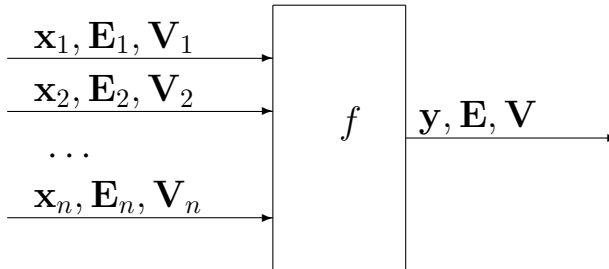
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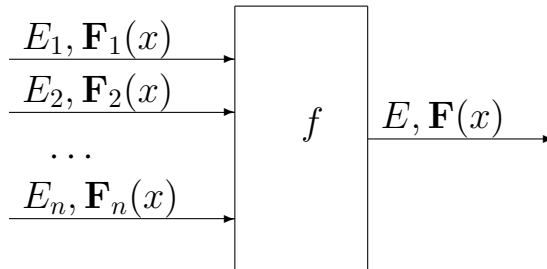
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45. Extension of Interval Arithmetic to Probabilistic Case: Challenges

- intervals + 2nd moments:



- moments + p-boxes; e.g.:



46. Case Study: Bioinformatics

- *Practical problem:* find genetic difference between cancer cells and healthy cells.
- *Ideal case:* we directly measure concentration c of the gene in cancer cells and h in healthy cells.
- *In reality:* difficult to separate.
- *Solution:* we measure $y_i \approx x_i \cdot c + (1 - x_i) \cdot h$, where x_i is the percentage of cancer cells in i -th sample.
- *Equivalent form:* $a \cdot x_i + h \approx y_i$, where $a \stackrel{\text{def}}{=} c - h$.

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47. Case Study: Bioinformatics (cont-d)

- *If we know x_i exactly:* Least Squares Method

$$\sum_{i=1}^n (a \cdot x_i + h - y_i)^2 \rightarrow \min_{a,h}, \text{ hence } a = \frac{C(x,y)}{V(x)} \text{ and}$$

$$h = E(y) - a \cdot E(x), \text{ where } E(x) = \frac{1}{n} \cdot \sum_{i=1}^n x_i,$$

$$V(x) = \frac{1}{n-1} \cdot \sum_{i=1}^n (x_i - E(x))^2,$$

$$C(x,y) = \frac{1}{n-1} \cdot \sum_{i=1}^n (x_i - E(x)) \cdot (y_i - E(y)).$$

- *Interval uncertainty:* experts manually count x_i , and only provide interval bounds \mathbf{x}_i , e.g., $x_i \in [0.7, 0.8]$.
- *Problem:* find the range of a and h corresponding to all possible values $x_i \in [\underline{x}_i, \bar{x}_i]$.

48. Extension of Interval Arithmetic to Probabilistic Case: General Problem

- *General problem:*

- we know intervals $\mathbf{x}_1 = [\underline{x}_1, \bar{x}_1], \dots, \mathbf{x}_n = [\underline{x}_n, \bar{x}_n]$,

- compute the range of $E(x) = \frac{1}{n} \sum_{i=1}^n x_i$, population

$$\text{variance } V = \frac{1}{n} \sum_{i=1}^n (x_i - E(x))^2, \text{ etc.}$$

- *Difficulty:* NP-hard even for variance.

- *Known:*

- efficient algorithms for \underline{V} ,

- efficient algorithms for \bar{V} and $C(x, y)$ for reasonable situations.

- *Bioinformatics case:* find intervals for $C(x, y)$ and for $V(x)$ and divide.

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49. Proof of Symmetry Result: Part 1

- Due to scale-invariance, for every $y_1, \dots, y_n, y'_1, \dots, y'_n$, we can take $\lambda_i = \frac{1}{y_i}$ and conclude that

$$(y'_1, \dots, y'_n) \sim (y_1, \dots, y_n) \Leftrightarrow \left(\frac{y'_1}{y_1}, \dots, \frac{y'_n}{y_n} \right) \sim (1, \dots, 1).$$

- Thus, to describe the equivalence relation \sim , it is sufficient to describe $\{z = (z_1, \dots, z_n) : z \sim (1, \dots, 1)\}$.
- Similarly,

$$(y'_1, \dots, y'_n) \succ (y_1, \dots, y_n) \Leftrightarrow \left(\frac{y'_1}{y_1}, \dots, \frac{y'_n}{y_n} \right) \succ (1, \dots, 1).$$

- Thus, to describe the ordering relation \succ , it is sufficient to describe the set $\{z = (z_1, \dots, z_n) : z \succ (1, \dots, 1)\}$.
- Similarly, it is also sufficient to describe the set

$$\{z = (z_1, \dots, z_n) : (1, \dots, 1) \succ z\}.$$

50. Proof of Symmetry Result: Part 2

- *To simplify:* take logarithms $Y_i = \ln(y_i)$, and sets

$$S_{\sim} = \{Z : z = (\exp(Z_1), \dots, \exp(Z_n)) \sim (1, \dots, 1)\},$$

$$S_{\succ} = \{Z : z = (\exp(Z_1), \dots, \exp(Z_n)) \succ (1, \dots, 1)\};$$

$$S_{\prec} = \{Z : (1, \dots, 1) \succ z = (\exp(Z_1), \dots, \exp(Z_n))\}.$$

- Since the pre-ordering relation is total, for Z , either $Z \in S_{\sim}$ or $Z \in S_{\succ}$ or $Z \in S_{\prec}$.

- *Lemma:* S_{\sim} is closed under addition:

- $Z \in S_{\sim}$ means $(\exp(Z_1), \dots, \exp(Z_n)) \sim (1, \dots, 1)$;
- due to scale-invariance, we have

$$(\exp(Z_1 + Z'_1), \dots) = (\exp(Z_1) \cdot \exp(Z'_1), \dots) \sim (\exp(Z'_1), \dots);$$

- also, $Z' \in S_{\sim}$ means $(\exp(Z'_1), \dots) \sim (1, \dots, 1)$;
- since \sim is transitive,

$$(\exp(Z_1 + Z'_1), \dots) \sim (1, \dots) \text{ so } Z + Z' \in S_{\sim}.$$

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51. Proof of Symmetry Result: Part 3

- *Reminder:* the set S_{\sim} is closed under addition;
- Similarly, S_{\succ} and S_{\prec} are closed under addition.
- *Conclusion:* for every integer $q > 0$:
 - if $Z \in S_{\sim}$, then $q \cdot Z \in S_{\sim}$;
 - if $Z \in S_{\succ}$, then $q \cdot Z \in S_{\succ}$;
 - if $Z \in S_{\prec}$, then $q \cdot Z \in S_{\prec}$.
- Thus, if $Z \in S_{\sim}$ and $q \in \mathbb{N}$, then $(1/q) \cdot Z \in S_{\sim}$.
- We can also prove that S_{\sim} is closed under $Z \rightarrow -Z$:
 - $Z = (Z_1, \dots) \in S_{\sim}$ means $(\exp(Z_1), \dots) \sim (1, \dots)$;
 - by scale invariance, $(1, \dots) \sim (\exp(-Z_1), \dots)$, i.e., $-Z \in S_{\sim}$.
- Similarly, $Z \in S_{\succ} \Leftrightarrow -Z \in S_{\prec}$.
- So $Z \in S_{\sim} \Rightarrow (p/q) \cdot Z \in S_{\sim}$; in the limit, $x \cdot Z \in S_{\sim}$.

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52. Proof of Symmetry Result: Final Part

- *Reminder:* S_{\sim} is closed under addition and multiplication by a scalar, so it is a linear space.
- *Fact:* S_{\sim} cannot have full dimension n , since then all alternatives will be equivalent to each other.
- *Fact:* S_{\sim} cannot have dimension $< n - 1$, since then:
 - we can select an arbitrary $Z \in S_{\succ}$;
 - connect it w/ $-Z \in S_{\succ}$ by a path γ that avoids S_{\sim} ;
 - due to closeness, $\exists \gamma(t^*)$ in the limit of S_{\succ} and S_{\prec} ;
 - thus, $\gamma(t^*) \in S_{\sim}$ – a contradiction.
- Every $(n - 1)$ -dim lin. space has the form $\sum_{i=1}^n \alpha_i \cdot Y_i = 0$.
- Thus, $Y \in S_{\succ} \Leftrightarrow \sum \alpha_i \cdot Y_i > 0$, and

$$y \succ y' \Leftrightarrow \sum \alpha_i \cdot \ln(y_i/y'_i) > 0 \Leftrightarrow \prod y_i^{\alpha_i} > \prod y'_i^{\alpha_i}.$$

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