MV–algebras and Intermediate Syllogisms

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Aristotle (384 - 322 BCE) introduced the study of quantification as a main part of the discipline of logic. His syllogisms can be seen as a formal study of the meaning of the four basic quantifier expressions all, no, some, not all and of their properties. Syllogisms combine Subjects are Predicates, e.g. All Scholars are Poor.

The second major historical contribution to the theory of quantifiers came from the inventor of modern logic, Frege, in the 1870s. He wanted to avoid the focus on Subject-Predicate form, which he saw as having been a major obstacle to the development of logic after Aristotle. It was therefore an important discovery that these quantifiers could all be defined in terms of $\forall$ and sentential connectives: replace

$$\text{all}(S, P) \text{ by } \quad \forall x(S(x) \texttt{ imp } P(x)),$$
$$\text{some}(S, P) \text{ by } \texttt{not } \forall x(S(x) \texttt{ imp not } P(x)).$$

We study generalized syllogisms introduced by Peterson in 2000. Our aim is to show how they are related to many valued logic. We need an algebraic structure called MV-algebra. This structure will be essential all over.

However, at this stage we only give a simple example of an MV-algebra (called standard MV-algebra). Assume $a, b \in [0, 1]$. Set

- $a \odot b = \max\{a + b - 1, 0\}$ (Łukasiewicz product),
- $a \oplus b = \min\{a + b, 1\}$ (Łukasiewicz sum),
- $a \to b = \max\{1 - a + b, 0\}$ (Łukasiewicz implication),
- $a^* = 1 - a$ (Łukasiewicz negation).

Then $a \odot b \leq c$ iff $a \leq b \to c$, in particular, $a \leq b$ iff $a \to b = 1$.

Notice that if $a, b \in \{0, 1\}$ we obtain a Boolean algebra.

Peterson's Linguistic Analysis on Generalized Aristotelian Syllogisms

- Aristotelian syllogisms deal with the phrases

1. All S are P (negation: No S are P) Universal
2. Some S are P (negation: Some S are not P) Particular

- In his book (2000), Peterson adds and analyzed three more phrases

1. Almost–all S are P (negation: Few S are P) Predominant
2. Most S are P (negation: Most S are not P) Majority
3. Many S are P (negation: Many S are not P) Common

- After Peterson's analysis, it is clear that in everyday language use quantifiers do not follow mathematical logic; for example All politicians are liers does not automatically entail Some politicians are liers if some is understood as some, but not all.

Syllogisms are grouped into the four Figures

Figure I

A quantity Q_1 of M are P	(premise 1)	Abbreviated	MP
A quantity Q_2 of S are M	(premise 2)		SM
A quantity Q_3 of S are P	(conclusion)		

Figure II

A quantity Q_1 of P are M	(premise 1)	Abbreviated	PM
A quantity Q_2 of S are M	(premise 2)		SM
A quantity Q_3 of S are P	(conclusion)		

Figure III

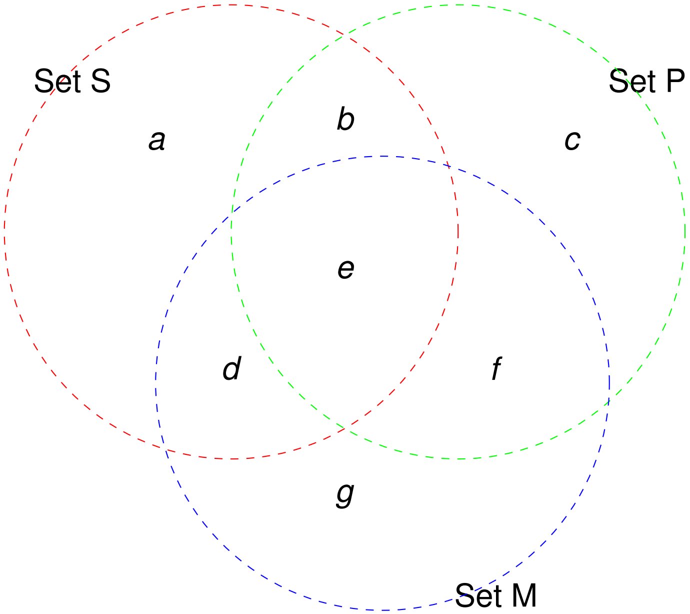
A quantity Q_1 of M are P	(premise 1)	Abbreviated	MP
A quantity Q_2 of M are S	(premise 2)		MS
A quantity Q_3 of S are P	(conclusion)		

Figure IV

A quantity Q_1 of P are M	(premise 1)	Abbreviated	PM
A quantity Q_2 of M are S	(premise 2)		MS
A quantity Q_3 of S are P	(conclusion)		

Syllogisms can be presented and valided/invalid by following Venn diagrams.

Principle of imported existence: empty (sub–)set are not considered.



Peterson's square			
Affirmative statements:		Negative statements:	Generic term
A: All X are Y	$\leftrightarrow$	**E**: All X are non–Y	universal
$\Downarrow$		$\Downarrow$	
P: Almost–all X are Y	$\leftrightarrow$	**B**: Almost–all X are non–Y	predominant
$\Downarrow$		$\Downarrow$	
T: Most X are Y	$\leftrightarrow$	**D**: Most X are non–Y	majority
$\Downarrow$		$\Downarrow$	
K: Many X are Y	$\cdots$	**G**: Many X are non–Y	common
$\Downarrow$		$\Downarrow$	
I: Some X are Y	$\cdots$	**O**: Some X are non–Y	particular

1. **A** and **E**, **P** and **B**, **T** and **D** are contrary pairs, denoted by **X** $\leftrightarrow$ **Y**. They can not be simultaniously true but can be simultaniously false.
2. The complement of **A**, **P**, and **T** is **E**, **B** and **D**, respectively. For universal, predominant and majority statements more than half of X is Y.
3. **K** and its complement **G** as well as **I** and its complement **O** compose sub–contrary pairs, denoted by '$\cdots$'. It is possible that 'Some X are Y' is true and simultaneously 'Some X are non–Y' is also true (however, it is not possible that both are false).

1. Statement pair (**A**, **O**), (**I**, **E**), (**P**, **G**) and (**K**, **B**) are pairs of contradictory statements. If the first one is true, then the second one is false and conversely. E.g., it is not possible that ‘Almost–all X are Y’ holds and simultaniously also ‘Many X are non–Y’ would hold.
2. We note here the following fact, the meaning of which Peterson has hardly noticed. If **P** and **G** are contradictory statements, i.e. in conflict with each other, then obviously so are (even more contradictory statements) **P** and **D** as well as **P** and **E**.
3. The $\Downarrow$ indicate immediate entailment: if e.g. All X are Y is true then also Almost–all X are Y is true (but not necessarily vice versa). This principle is to generalize the classical state of affairs; it is also closely connected to the semantical meaning of the quantifiers. Similarly to the classical **I**: Some X are Y which means that at least one or more X is Y, a statement **T**: Most X are Y is to be interpreted Most X or more are Y.
4. Statement **A** is stronger than statement **P**, which is stronger than statement **T**, etc, and equivalently, statement **I** is weaker than statement **K** etc. Similarly for negative statements. Thus, the meaning of phrases strengthen a premise and weaken a conclusion should be clear.

Based on a profound and detailed study on the English quantifiers Almost–all, Most and Many, Peterson sums up several principles related to intermediate syllogisms; we will need the following:

1. At least one premise must be affirmative.
2. The conclusion is negative if, and only if one of the premises is negative.
3. At least one of the premise must have a quantity of preponderance.
4. If any premise is non–universal, then the conclusion must have a quantity that is less than or equal to that premise.

Intermediate syllogisms are presented briefly by XYZ–F, for example KAI–III refers to the following syllogism in Figure III

Many M are P	(premise 1 is a statement of form K)
All M are S	(premise 2 is a statement of form A)
Some S are P	(conclusion is a statement of form I)

Valid intermediate syllogisms in 5 quantities approach

On the bases of such analysis and notions, Peterson introduces a special (new) Venn Diagram method for validating intermediate syllogisms. This method is quite complex and rather unintuitive. Finally, 105 syllogisms of all 4000 are valid. Aristotelian (classical) syllogisms are printed by bold font. Notice that all the intermediate syllogisms ‘fall in between the classical ones’.

Figure I	Affirmative			
AAA				
AAP	APP			
AAT	APT	ATT		
AAK	APK	ATK	AKK	
AAI	API	ATI	AKI	**AII**

Example: AKK–I

All rabbits have fur.
Many pets are rabbits.

Many pets have fur.

Example: ATK–I

All women are mortal.
Most Greeks are women.

Many Greeks are mortal.

Example: APP–I

All humans are mortal.
Almost–all Swedish are humans.

Almost–all Swedish are mortal.

Figure I, negative

Figure I	Negative			
EAE				
EAB	EPB			
EAD	EPD	ETD		
EAG	EPG	ETG	EKG	
EAO	EPO	ETO	EKO	**EIO**

Example: ETG–I

No homework is fun.
Most reading is homework.

Many reading is not fun.

Figure II, negative case 1

Figure II	Negative	Case 1		
AEE				
AEB	ABB			
AED	ABD	ADD		
AEG	ABG	ADG	AGG	
AEO	ABO	ADO	AGO	**AOO**

Example: ADD–II

All informative things are useful.
Most TV–soaps are not useful.

Most TV–soaps are not informative.

Figure II, negative case 2

Figure II	Negative	Case 2		
EAE				
EAB	EPB			
EAD	EPD	ETD		
EAG	EPG	ETG	EKG	
EAO	EPO	ETO	EKO	**EIO**

Example: EKG–II

No methyl alcohol–containing beverage is healthy.
Many liquors are healthy.

Many liquor contain no methyl alcohol.

Figure III, affirmative

Figure III	Affirmative			
AAI	PAI	TAI	KAI	**IAI**
API	PPI	TPI	KPI	
ATI	PTI	TTI		
AKI	PKI			
AII				

Example: TTI–III

Most jokes are old.
Most jokes are funny.

Some funny jokes are old.

Notice that KKI–3 is not valid!

Many jokes are old.
Many jokes are funny.

Some funny jokes are old.

Figure III, negative

Figure III	Negative			
EAO	BAO	DAO	GAO	**OAO**
EPO	BPO	DPO	GPO	
ETO	BTO	DTO		
EKO	BKO			
EIO				

Example: GPO–III

Many mathematicians are not ballet dancers.
Almost–all mathematicians are quiet persons.

Some quiet persons are not ballet dancers.

Figure IV

Figure IV		
AAI	**AEE**	**EAO**
PAI	AEB	EPO
TAI	AED	ETO
KAI	AEG	EKO
IAI	**AEO**	**EIO**

Example: EKO–IV

No dogs are husbands.
Many huspands are pets.

Some pets are not dogs.

Observations on Peterson's 5 quantities approach

Contradictory pairs (**X**, **Y**) are closely connected with valid syllogisms in Figure III. It is enough to focus on such contradictory pairs (**X**, **Y**) that **X** is an affirmative preponderance statement and **Y** is a negative non–preponderance statement. Next focus on (**P**, **G**). It is a border line contradictory pair. This leads us associate with each statement **S** a grade $q \in (0, 1]$:

Affirmative	Grade	Negative	Grade
A	1	**E**	0
P	p	**B**	$1 - p$
T	t	**D**	$1 - t$
K	k	**G**	$1 - k$
I	ϵ	**O**	$1 - \epsilon$

1. $0 < \epsilon < 1 - p < k < \frac{1}{2} < t < p < 1$,
2. (b) $k + p > 1$; (**P**, **G**) is (a border line) intermediate contradictory pair,
3. $t + k \leq 1$; (**K**, **D**) is not a contradictory pair.

Valid syllogisms in Figure I: we observe the following

1. The first premise in each of the negative syllogisms in Figure I is the classical **E**: 'All S are non–P'.
2. Denote the grade of the second premise by $W (= 1, \epsilon$ or some $q)$ and the grade of the conclusion by $U (= 0, 1 - \epsilon$ or some $1 - q)$. Then the corresponding syllogism is valid if, and only if $W \oplus U = 1$, where $\oplus$ is the Łukasiewicz sum. Indeed, see the following Cayley table:

$\oplus$	(**E**, 0)	(**B**, $1-p$)	(**D**, $1-t$)	(**G**, $1-k$)	(**O**, $1-\epsilon$)
(**A**, 1)	1	1	1	1	1
(**P**, p)	< 1	1	1	1	1
(**T**, t)	< 1	< 1	1	1	1
(**K**, k)	< 1	< 1	< 1	1	1
(**I**, ϵ)	< 1	< 1	< 1	< 1	1

1. The first premise in each of the affirmative syllogisms in Figure I is the classical **A** 'All S are P'.
2. Denote the grade of the second premise by $W (= 1, \epsilon$ or some $q)$ and the grade of the conclusion by $U (= 1, \epsilon$ or some $q)$. Then the corresponding syllogism is valid if, and only if $U^* \oplus W = 1$.

Valid syllogisms in Figure II: we observe the following

All valid syllogisms in Figure II are negative. Figure II is obtained from Figure I by changing places of P and M with each other in the first premise. Consequently, a large symmetry follows. Indeed, the negative Case 2 in Figure II has exactly the same valid syllogisms that the negative syllogisms in Figure I. Moreover, we observe that

1. The first premise in each of the syllogisms in negative Case 2 is the classical **E** 'All S are non–P'.
2. Denote the grade of the second premise (affirmative) by $W(=1, \epsilon$ or some $q)$ and the grade of the conclusion by $U(=0, 1-\epsilon$ or some $1-q)$. Then the corresponding syllogism is valid if, and only if $W \oplus U = 1$.

The valid syllogisms in the negative Case 1 in Figure II are obtained by substituting both premises by their complements in Case 2. Moreover, if we denote the grade of the second premise (negative) by $W(=0, 1-\epsilon$ or some $1-q)$ and the grade of the conclusion by $U(=0, 1-\epsilon$ or some $1-q)$. Then the corresponding syllogism is valid if, and only if $W^* \oplus U = 1$.

Valid syllogisms in Figure IV: we observe the following

1. All the intermediate valid syllogisms fall in between pairs of tradional ones, e.g. TAI falls in between AAI and IAI.
2. An intermediate syllogism in Figure IV is valid if, and only if it is obtained from a classical syllogism by replacing one of the premises by a stronger one or the conclusion by a weaker one, where the notions of stronger and weaker are presented in Peterson's square by the symbol $\Downarrow$.

For example AEG is obtained by AED by replacing the conclusion D by a weaker G and ETO is obtained from EKO by replacing the second premise K by a stronger T.

Theorem (Valid affirmative syllogisms in Figure III)

An affirmative syllogism in Figure III is valid and the conclusion is **I** *'Some S are P' if $V \odot W \neq 0$, where V and W are the grades associated with first and second premise, respectively. Here $\odot$ is the Łukasiewicz product. All other affirmative syllogism in Figure III are invalid.*

Proof. Consider the following Cayley table:

$\odot$	$(A, 1)$	(P, p)	(T, t)	(K, k)	(I, ϵ)
$(A, 1)$	> 0	> 0	> 0	> 0	> 0
(P, p)	> 0	> 0	> 0	> 0	0
(T, t)	> 0	> 0	> 0	0	0
(K, k)	> 0	> 0	0	0	0
(I, ϵ)	> 0	0	0	0	0

This proves the claim.

Theorem (Valid negative syllogisms in Figure III)

A negative syllogism in Figure III is valid with conclusion **O** *'Some S are non–P' if* $V^* \odot W \neq 0$*, where* V^* *and W are the grades associated with the complement of the first and to the second premise, respectively. All other negative syllogism in Figure III are invalid.*

Proof. Consider the following Cayley table:

	$\odot$	$(A,1)$	(P,p)	(T,t)	(K,k)	(I,ϵ)
$\neg(E,0) =$	$(A,1)$	>0	>0	>0	>0	>0
$\neg(B,1-p) =$	(P,p)	>0	>0	>0	>0	0
$\neg(D,1-t) =$	(T,t)	>0	>0	>0	0	0
$\neg(G,1-k) =$	(K,k)	>0	>0	0	0	0
$\neg(O,1-\epsilon) =$	(I,ϵ)	>0	0	0	0	0

The proof is complete.

Syllogisms with $k > 2$ quantities

To generalize Peterson's results, the following is obvious.

1. An evident starting point for extending Peterson's system is to linguistically analyse new quantities and their relation to the old ones. This extends Peterson's square.
2. Introducing a new quantity should leave the 105 intermediate syllogisms untouched. For example, KKI–III is invalid, thus, it should remain invalid in all extended systems of syllogisms, similarly, PPI–3 is valid, so it should remain valid in all extensions. If an intermediate syllogism is 'an empirical fact', then its validity does not depend on the other quantifiers we have. In short, any generalization has to be conservative.
3. The order (mutual strength) of statements and contradictory pairs are important.
4. Unfortunately Peterson's approach of fractional quantifiers is not concervative.

Fractional syllogistic systems are not concervative

Figure III	Affirmative	(+'Half'	+'Couple')			
AAI	PAI	TAI	FAI	KAI	MAI	IAI
API	PPI	TPI	FPI	KPI	MPI?	
ATI	PTI	TTI	FTI	KTI×		
AFI	PFI	TFI	FFI×			
AKI	PKI	TKI×				
AMI	PMI?					
AII						

However, TKI–III, KTI–III and FFI–III are invalid. Are

Almost–all M are P
Only a couple of M are S
―――
Some S are P

and

Only a couple of M are P
Almost–all M are S
―――
Some S are P

valid intermediate syllogisms? There is no pure mathematical answer, rather the answer is related to the question: are (**P**, **N**) and (**M**, **B**) pairs of contradictory statements? The question needs to be solved by a linguistic research. Peterson's claim of 28($= \frac{7^2+7}{2}$) valid syllogisms is incorrect; there are only 23 or 25.

An example of extended Peterson's square

The logic of Some, Just a couple, Many, Half, Most, Almost–all, All with the exception of a few pathological cases and All. (**S**,**G**) constitute a border line contradictory pair, i.e (**S**,**N**) is not contradictory pair, while (**P**,**G**) is.

Extended Peterson's square		
Affirmative statements	$q \leftrightarrow 1 - q$	Negative statements:
A: All X are Y	$1.0 \leftrightarrow 0.0$	**E**: All X are non–Y
$\Downarrow$		$\Downarrow$
S: All–but X are Y	$0.9 \leftrightarrow 0.1$	**Z**: All–but X are non–Y
$\Downarrow$		$\Downarrow$
P: Almost–all X are Y	$0.8 \leftrightarrow 0.2$	**B**: Almost–all X are non–Y
$\Downarrow$		$\Downarrow$
T: Most X are Y	$0.6 \leftrightarrow 0.4$	**D**: Most X are non–Y
$\Downarrow$		$\Downarrow$
F: Half X are Y	$0.5 \cdots 0.5$	**V**: Half X are non–Y
$\Downarrow$		$\Downarrow$
K: Many X are Y	$0.4 \cdots 0.6$	**G**: Many X are non–Y
$\Downarrow$		$\Downarrow$
M: Couple X are Y	$0.05 \cdots 0.95$	**N**: Couple X are non–Y
$\Downarrow$		$\Downarrow$
I: Some X are Y	$0.01 \cdots 0.99$	**O**: Some X are non–Y

An example: Valid affirmative syllogisms in Figure III

Valid affirmative syllogisms in Figure III are those that satisfy condition $V \odot W \neq 0$;

AAI	SAI	PAI	TAI	FAI	KAI	MAI	**IAI**
ASI	SSI	PSI	TSI	FSI	KSI	X	
API	SPI	PPI	TPI	FPI	KPI		
ATI	STI	PTI	TTI	FTI			
AFI	SFI	PFI	TFI				
AKI	SKI	PKI					
AMI	X						
AII							

Conclusion

1. Peterson's intermediate syllogisms extend Aristotelian syllogisms by accepting three intermediate categorical propositions. The 105 valid intermediate syllogisms are an empirical fact of correct reasoning, not a theory of possible way of reasoning.
2. We do not define any new theory on Peterson's intermediate syllogisms; we only demonstrate that, by associating certain values V, W and U on standard Łukasiewicz algebra with the first and second premise and the conclusion, respectively, the validity of the corresponding intermediate syllogism is determined by a simple MV–algebra equation.
3. We discuss extensions of Peterson's system. Due to the empirical nature of the 105 valid intermediate syllogisms including the original 24 Aristotelian, a proper extension or restriction must be conservative in a sense that validity or invalidity of any existing syllogism must remain in this extension or restriction. In this respect our approach differs from Peterson's fractional syllogisms. An extension must be based on a linguistic analysis of new quantifiers, their relation to the existing quantifiers, and pairs of contradictory intermediate categorical propositions.

Exercises.

1° Calculate Łukasiewicz product, sum, implication and negation for $a = 0.9, b = 0.8$.

2° Prove that De Morgan law $a \odot b = (a^* \oplus b^*)^*$ holds for all $a, b \in [0, 1]$.

3° Study the following generalized syllogism.

Many mushrooms are not tasty.
Almost–all mushrooms are edible natural issues.

Some edible natural issue are not tasty.

a) To which Figure does it belong and b) is it valid?