# Paraconsistent Pavelka logic

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The contemporary logical orthodoxy has it that, from contradictory premises, anything can be inferred. To be more precise, let $\models$ be a relation of logical consequence, defined either semantically or proof–theoretically. Call $\models$ explosive if it validates $\{A, \neg A\} \models B$ for every $A$ and $B$ (ex contradictione quodlibet).

The contemporary orthodoxy, i.e., classical logic, is explosive, but also some non-classical logics such as intuitionist logic and most other standard logics are explosive.

The major motivation behind paraconsistent logic is to challenge this orthodoxy. A logical consequence relation, $\models$, is said to be paraconsistent if it is not explosive. Thus, if $\models$ is paraconsistent, then even if we are in certain circumstances where the available information is inconsistent, the inference relation does not explode into triviality. Thus, paraconsistent logic accommodates inconsistency in a sensible manner that treats inconsistent information as informative.

In Belnap's paraconsistent logic, four possible values associated with atomic formulas $\alpha$ are interpreted as told only True, told only False, both told True and told False and neither told True nor told False, respectively.

However, we call them for simplicity true, false, contradictory and unknown: if there is evidence for $\alpha$ and no evidence against $\alpha$, then $\alpha$ obtains the value true and if there is no evidence for $\alpha$ and evidence against $\alpha$, then $\alpha$ obtains the value false. A value contradictory corresponds to a situation where there is simultaneously evidence for $\alpha$ and against $\alpha$ and, finally, $\alpha$ is labeled by value unknown if there is no evidence for $\alpha$ nor evidence against $\alpha$.

More formally, the values are associated with ordered couples $\mathbb{T} = \langle 1, 0 \rangle$, $\mathbb{F} = \langle 0, 1 \rangle$, $\mathbb{K} = \langle 1, 1 \rangle$ and $\mathbb{U} = \langle 0, 0 \rangle$, respectively.

Belnap's ideas can be generalized to a continuous valued logic in the following way. Associate to a formula $\alpha$ an ordered couple $\langle a, b\rangle$, called evidence couple, where $a, b \in [0,1]$. The intuitive meaning of $a$ and $b$ is the degree of evidence for a statement $\alpha$ and against $\alpha$, respectively. Graded values on $[0,1]$, computed via

$$t(\alpha) = \min\{a, 1-b\}, \quad (1)$$
$$k(\alpha) = \max\{a+b-1, 0\}, \quad (2)$$
$$u(\alpha) = \max\{1-a-b, 0\}, \quad (3)$$
$$f(\alpha) = \min\{1-a, b\}, \quad (4)$$

mean t($\alpha$) = truth, k($\alpha$) = contradictory, u($\alpha$) = unknown and f($\alpha$) = falsehood of the statement $\alpha$. Truth and falsehood are not each others complements as $f(\alpha)+k(\alpha)+u(\alpha)+t(\alpha)=1$. Moreover, the set $\mathcal{M}$ of $2\times 2$ evidence matrices $M$ are of a form

$$M = \begin{bmatrix} f(\alpha) & k(\alpha) \\ u(\alpha) & t(\alpha) \end{bmatrix} = \begin{bmatrix} a^* \wedge b & a \odot b \\ a^* \odot b^* & a \wedge b^* \end{bmatrix}$$

The main question now is: given a complete MV-algebra $\mathbf{L}$, can we equip the set

$$\mathcal{M} = \left\{ \begin{bmatrix} a^* \wedge b & a \odot b \\ a^* \odot b^* & a \wedge b^* \end{bmatrix} | \langle a, b \rangle \in L \times L \right\}$$

with such binary operations that we obtain a complete MV-algebra? The answer is affirmative as we have the following three theorems

PROPOSITION 1 *In an MV–algebra* $\mathbf{L}$ *the following holds for all* $x, y \in \mathbf{L}$

$$(x \odot y) \wedge (x^* \odot y^*) = \mathbf{0}, \tag{5}$$

$$(x^* \wedge y) \oplus (x \odot y) \oplus (x^* \odot y^*) \oplus (x \wedge y^*) = \mathbf{1}. \tag{6}$$

Proposition 2 *Assume $x, y, a, b$ are elements of an MV–algebra* **L** *such that the following system of equations holds*

$$(A)\begin{cases} x^* \wedge y & = & a^* \wedge b, \\ x \odot y & = & a \odot b, \\ x^* \odot y^* & = & a^* \odot b^*, \\ x \wedge y^* & = & a \wedge b^*. \end{cases}$$

*Then $x = a$ and $y = b$.*

Proposition 3 *Assume $x, y$ are elements of an MV–algebra* **L** *such that*

$$(B)\begin{cases} x^* \wedge y & = & f, \\ x \odot y & = & k, \\ x^* \odot y^* & = & u, \\ x \wedge y^* & = & t. \end{cases}$$

*Then (C) $x = t \oplus k$, $y = f \oplus k$ and (D) $x = (f \oplus u)^*$, $y = (t \oplus u)^*$.*

Proposition 2 and Proposition 3 put ordered couples $\langle x, y \rangle$ and values $f, k, u, t$ defined by (B) into a one–one correspondence.

Let $\mathbf{L} = \langle L, \oplus, ^*, \mathbf{0} \rangle$ be an MV–algebra. The product set $L \times L$ can be equipped with an MV–structure by setting

$$\langle a, b \rangle \otimes \langle c, d \rangle = \langle a \oplus c, b \odot d \rangle, \quad (7)$$

$$\langle a, b \rangle^{\perp} = \langle a^*, b^* \rangle, \quad (8)$$

$$\overline{\mathbf{0}} = \langle \mathbf{0}, \mathbf{1} \rangle \quad (9)$$

for each ordered couple $\langle a, b \rangle, \langle c, d \rangle \in L \times L$. The order on $L \times L$ is defined via

$$\langle a, b \rangle \leq \langle c, d \rangle \text{ if and only if } a \leq c, d \leq b, \quad (10)$$

the lattice operation by

$$\langle a, b \rangle \vee \langle c, d \rangle = \langle a \vee c, b \wedge d \rangle, \quad (11)$$

$$\langle a, b \rangle \wedge \langle c, d \rangle = \langle a \wedge c, b \vee d \rangle, \quad (12)$$

and an adjoint couple $\langle \star, \mapsto \rangle$ by

$$\langle a, b \rangle \star \langle c, d \rangle = \langle a \odot c, b \oplus d \rangle, \quad (13)$$

$$\langle a, b \rangle \mapsto \langle c, d \rangle = \langle a \to c, (d \to b)^* \rangle. \quad (14)$$

Given an MV-algebra **L**, denote the structure described via (7) - (14) by $\mathbf{L}_{EC}$ and call it the MV–algebra of evidence couples induced by **L** and call the set

$$\mathcal{M} = \left\{ \begin{bmatrix} a^* \wedge b & a \odot b \\ a^* \odot b^* & a \wedge b^* \end{bmatrix} | \langle a, b \rangle \in L \times L \right\}$$

the set of evidence matrices induced by evidence couples. By Proposition 2 and Proposition 3 we have

**Theorem** There is a one–to–one correspondence between $L \times L$ and $\mathcal{M}$: if $A, B \in \mathcal{M}$ are two evidence matrices induced by evidence couples $\langle a, b \rangle$ and $\langle x, y \rangle$, respectively, then $A = B$ if and only if $a = x$ and $b = y$.

The MV–structure descends from $\mathbf{L}_{EC}$ to $\mathcal{M}$ in a natural way: if $A, B \in \mathcal{M}$ are two evidence matrices induced by evidence couples $\langle a, b\rangle$ and $\langle x, y\rangle$, respectively, then the evidence couple $\langle a \oplus x, b \odot y\rangle$ induces an evidence matrix

$$C = \begin{bmatrix} (a \oplus x)^* \wedge (b \odot y) & (a \oplus x) \odot (b \odot y) \\ (a \oplus x)^* \odot (b \odot y)^* & (a \oplus x) \wedge (b \odot y)^* \end{bmatrix}.$$

Thus, we may define a binary operation $\bigoplus$ on $\mathcal{M}$ by

$$\begin{bmatrix} a^* \wedge b & a \odot b \\ a^* \odot b^* & a \wedge b^* \end{bmatrix} \bigoplus \begin{bmatrix} x^* \wedge y & x \odot y \\ x^* \odot y^* & x \wedge y^* \end{bmatrix} = C.$$

Similarly, if $A \in \mathcal{M}$ is an evidence matrix induced by an evidence couple $\langle a, b\rangle$, then the evidence couple $\langle a^*, b^*\rangle$ induces an evidence matrix

$$A^{\perp} = \begin{bmatrix} a \wedge b^* & a^* \odot b^* \\ a \odot b & a^* \wedge b \end{bmatrix}.$$

In particular, the evidence couple $\langle \mathbf{0}, \mathbf{1} \rangle$ induces the following evidence matrix

$$\mathbb{F} = \begin{bmatrix} \mathbf{0}^* \wedge \mathbf{1} & \mathbf{0} \odot \mathbf{1} \\ \mathbf{0}^* \odot \mathbf{1}^* & \mathbf{0} \wedge \mathbf{1}^* \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Moreover, it is easy to verify that the evidence couples $\langle \mathbf{1}, \mathbf{0} \rangle$, $\langle \mathbf{1}, \mathbf{1} \rangle$ and $\langle \mathbf{0}, \mathbf{0} \rangle$ induce the following evidence matrices

$$\mathbb{T} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}, \mathbb{K} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathbb{U} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{bmatrix},$$

respectively. The matrices $\mathbb{F}, \mathbb{T}, \mathbb{K}, \mathbb{U}$ correspond to Belnap's original values false, true, contradictory, unknown, respectively.

Now we can write the following important

**Theorem** Let **L** be a complete MV–algebra. Then $\mathcal{M} = \langle \mathcal{M}, \bigoplus, ^{\perp}, F \rangle$ as defined above is a complete MV-algebra, called the MV–algebra of evidence matrices. Moreover, if

$$A = \begin{bmatrix} a^* \wedge b & a \odot b \\ a^* \odot b^* & a \wedge b^* \end{bmatrix}, \; B = \begin{bmatrix} x^* \wedge y & x \odot y \\ x^* \odot y^* & x \wedge y^* \end{bmatrix} \in \mathcal{M}$$

Then it the lattice operations $\wedge$, $\vee$, the monoidal operation $\bigodot$ and the residual operation $\longrightarrow$ are defined via

$$A \wedge B = \begin{bmatrix} (a \wedge x)^* \wedge (b \vee y) & (a \wedge x) \odot (b \vee y) \\ (a \wedge x)^* \odot (b \vee y)^* & (a \wedge x) \wedge (b \vee y)^* \end{bmatrix},$$

$$A \vee B = \begin{bmatrix} (a \vee x)^* \wedge (b \wedge y) & (a \vee x) \odot (b \wedge y) \\ (a \vee x)^* \odot (b \wedge y)^* & (a \vee x) \wedge (b \wedge y)^* \end{bmatrix},$$

$$A \bigodot B = \begin{bmatrix} (a \odot x)^* \wedge (b \oplus y) & (a \odot x) \odot (b \oplus y) \\ (a \odot x)^* \odot (b \oplus y)^* & (a \odot x) \wedge (b \oplus y)^* \end{bmatrix},$$

$$A \longrightarrow B = \begin{bmatrix} (a \to x)^* \wedge (y \to b)^* & (a \to x) \odot (y \to b)^* \\ (a \to x)^* \odot (y \to b) & (a \to x) \wedge (y \to b) \end{bmatrix}.$$

The above mathematical results show that we can develop a Pavelka style logic whose 'truth values' are evidence matrices. We obtain a many valued logic which, looking from outside is a consistent logic, but looking from inside - into the matrices - is paraconsistent. The obtained continuous valued paraconsistent logic is a complete logic in the Pavelka sense.

Before studying it in more detail, let us see some examples about evidence matrices.

Let **L** be the standard MV-algebra. Assume $\alpha$ and $\beta$ are associated with evidence couples $\langle 0.8, 0.4\rangle$ and $\langle 0.7, 0.2\rangle$, respectively. Then

$$v(\alpha) = \begin{bmatrix} 0.2 & 0.2 \\ 0 & 0.6 \end{bmatrix} \quad , \quad v(\texttt{not}\,\alpha) = \begin{bmatrix} 0.6 & 0 \\ 0.2 & 0.2 \end{bmatrix}$$

$$v(\beta) = \begin{bmatrix} 0.2 & 0 \\ 0.1 & 0.7 \end{bmatrix} \quad , \quad v(\texttt{not}\,\beta) = \begin{bmatrix} 0.7 & 0.1 \\ 0 & 0.2 \end{bmatrix}$$

$$v(\alpha \texttt{ and } \beta) = \begin{bmatrix} 0.5 & 0.1 \\ 0 & 0.4 \end{bmatrix} \quad , \quad v(\alpha \texttt{ or } \beta) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (= T)$$

$$v(\alpha \texttt{ imp } \beta) = \begin{bmatrix} 0 & 0 \\ 0.1 & 0.9 \end{bmatrix} \quad , \quad v(\alpha \texttt{ imp not}\,\beta) = \begin{bmatrix} 0.4 & 0 \\ 0.1 & 0.5 \end{bmatrix}$$

$$v(\beta \texttt{ imp } \alpha) = \begin{bmatrix} 0 & 0.2 \\ 0 & 0.8 \end{bmatrix} \quad , \quad v(\beta \texttt{ imp not}\,\alpha) = \begin{bmatrix} 0.4 & 0 \\ 0.1 & 0.5 \end{bmatrix}$$

$$v(\alpha \texttt{ equiv } \beta) = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.8 \end{bmatrix} \quad , \quad v(\alpha \texttt{ xor } \beta) = \begin{bmatrix} 0.4 & 0 \\ 0.1 & 0.5 \end{bmatrix}$$

$$v(\overline{\texttt{not}}\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad v(\overline{\texttt{not}}\beta) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$v(\alpha \ \overline{\texttt{and}} \ \beta) = \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0.6 \end{bmatrix} \quad , \quad v(\alpha \ \overline{\texttt{or}} \ \beta) = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.8 \end{bmatrix}$$

$$v(\alpha \ \overline{\texttt{imp}} \ \beta) = \begin{bmatrix} 0 & 0 \\ 0.3 & 0.7 \end{bmatrix} \quad , \quad v(\beta \ \overline{\texttt{imp}} \ \alpha) = \begin{bmatrix} 0 & 0.4 \\ 0 & 0.6 \end{bmatrix}$$

$$v(\alpha \ \overline{\texttt{equiv}} \ \beta) = \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0.6 \end{bmatrix},$$

where $\overline{\texttt{and}}$, $\overline{\texttt{imp}}$, etc are intuitionistic connectives; they are definable in Pavelka logic.

To illustrate the use of paraconsistent Pavelka logic, assume we have a standard MV-algebra valued fuzzy theory $\mathcal{T}$ with the following four non–logical axioms and evidence couples:

| Statement | formally | evidence |
|---|---|---|
| (1) If wages rise or prices rise there will be inflation | $(p$ or $q)$ imp $r$ | $\langle 1, 0 \rangle$ |
| (2) If there will be inflation, the Government will stop it or people will suffer | $r$ imp $(s$ or $t)$ | $\langle 0.9, 0.1 \rangle$ |
| (3) If people will suffer the Government will lose popularity | $t$ imp $w$ | $\langle 0.8, 0.1 \rangle$ |
| (4) The Government will not stop inflation and will not lose popularity | not $s$ and not $w$ | $\langle 1, 0 \rangle$ |

1° We show that $\mathcal{T}$ is satisfiable and therefore consistent. By **??** it is enough to consider evidence couples; focus on the following

| Statement | Atomic formula | Evidence couple |
|---|---|---|
| Wages rise | p | $\langle 0.3, 0.8 \rangle$ |
| Prices rise | q | $\langle 0, 1 \rangle$ |
| There will be inflation | r | $\langle 0.3, 0.8 \rangle$ |
| Government will stop inflation | s | $\langle 0, 1 \rangle$ |
| People will suffer | t | $\langle 0.2, 0.9 \rangle$ |
| Government will lose popularity | w | $\langle 0, 1 \rangle$ |

By direct computation we realize that they lead to the same evidence couples as in the fuzzy theory $\mathcal{T}$. Indeed, for example the evidence for the first non–logical axiom $[(p \texttt{ or } q) \texttt{ imp } r]$ is $(0.3 \oplus 0) \to 0.3 = 1$ and evidence against the axiom $[(p \texttt{ or } q) \texttt{ imp } r]$ is $(0.8 \odot 1)^* \odot 0.8 = 0$. Similarly for the other axioms. Thus, $\mathcal{T}$ is satisfiable and consistent.

2° What can be said on logical grounds about the claim wages will not rise, formally expressed by $\texttt{not}\, p$? The above consideration on evidence couples associates with $(\texttt{not}\, p)$ an evidence couple $\langle 0.3, 0.8 \rangle^{\perp} = \langle 0.7, 0.2 \rangle$ and the corresponding valuation $v$ is given by the evidence matrix

$$v(\texttt{not}\, p) = \begin{bmatrix} 0.7^* \wedge 0.2 & 0.7 \odot 0.2 \\ 0.7^* \odot 0.2^* & 0.7 \wedge 0.2^* \end{bmatrix} = \begin{bmatrix} 0.2 & 0 \\ 0.1 & 0.7 \end{bmatrix},$$

and the degree of tautology of $(\texttt{not}\, p)$ is less than or equal to $v(\texttt{not}\, p)$.

3° We prove that the degree of tautology of the wff $(\texttt{not}\, p)$ cannot be less that $v(\texttt{not}\, p)$, thus it is equal to $v(\texttt{not}\, p)$. To this end consider the following $\mathcal{R}$-proof:

| | | | |
|---|---|---|---|
| (1) | $(p \texttt{ or } q) \texttt{ imp } r$ | $\langle 1, 0\rangle$ | special axiom |
| (2) | $r \texttt{ imp } (s \texttt{ or } t)$ | $\langle 0.9, 0.1\rangle$ | special axiom |
| (3) | $t \texttt{ imp } w$ | $\langle 0.8, 0.1\rangle$ | special axiom |
| (4) | $\texttt{not}\, s \texttt{ and } \texttt{not}\, w$ | $\langle 1, 0\rangle$ | special axiom |
| (5) | $\texttt{not}\, w$ | $\langle 1, 0\rangle$ | (4), GS2 |
| (6) | $\texttt{not}\, s$ | $\langle 1, 0\rangle$ | (4), GS1 |
| (7) | $\texttt{not}\, t$ | $\langle 0.8, 0.1\rangle$ | (5), (3), GMTT |
| (8) | $\texttt{not}\, s \texttt{ and } \texttt{not}\, t$ | $\langle 0.8, 0.1\rangle$ | (6), (7), RBC |
| (9) | $\texttt{not}(s \texttt{ or } t)$ | $\langle 0.8, 0.1\rangle$ | (8), GDeM1 |
| (10) | $\texttt{not}\, r$ | $\langle 0.7, 0.2\rangle$ | (9), (2), GMTT |
| (11) | $\texttt{not}(p \texttt{ or } q)$ | $\langle 0.7, 0.2\rangle$ | (10), (1) GMTT |
| (12) | $\texttt{not}\, p \texttt{ and } \texttt{not}\, q$ | $\langle 0.7, 0.2\rangle$ | (11), GDeM2 |
| (13) | $\texttt{not}\, p$ | $\langle 0.7, 0.2\rangle$ | (12), GS1 |

4° By completeness of $\mathcal{T}$ we conclude

$$\mathcal{C}^{sem}(T)(\texttt{not}\, p) = \mathcal{C}^{syn}(T)(\texttt{not}\, p) = \left[\begin{array}{cc} 0.2 & 0 \\ 0.1 & 0.7 \end{array}\right].$$

We interpret this result by saying that, from a logical point of view, the claim wages will not rise is (much) more true than false, is not contradictory but lacks some information.

Exercise.
6° Assume $\alpha$ and $\beta$ are associated with evidence couples $\langle 0.9, 0.2\rangle$ and $\langle 0.6, 0.1\rangle$. What are the corresponding evidence matrices of $\alpha$, $\beta$, $\texttt{not}\,\alpha$, $\texttt{not}\,\beta$, $\alpha\,\texttt{and}\,\beta$, $\alpha\,\texttt{or}\,\beta$, $\alpha\,\texttt{imp}\,\beta$, $\alpha\,\texttt{equiv}\,\beta$?