Multivalued logic: historical notes

Francesc Esteva
(IIIA - CSIC)

Future contingent: Lukasiewicz text (1920)

I can assume without contradiction that my presence in Warsaw at noon on 21 December of next year is at present time determined neither positively nor negatively. It is possible but not necessary, that I shall be present in Warsaw at the given time. On this assumption the proposition "I shall be in Warsaw at noon on 21 Decembert of next year" can at the present time be neither true nor false. For if it where true, my future presence in W. would have to be necessary, which is contradictory to the assumption. If it where false now my future presence in W. would be impossible, which is also contradictory with the assumption. Therefore the proposition considered is at present moment neither true nor false and must posses a third value. It represents "the possible", and joins "the true" and "the false" as a third value. The three valued system of propositional logic owes its origin to this line of thought.

Lukasiewicz three-valued system

He used $\neg$ and $\longrightarrow$ as primitive connectives and define:
"OR" : $p \vee q$ as $(p \longrightarrow q) \longrightarrow q = (q \longrightarrow p) \longrightarrow p$
"AND": $p \wedge q$ as $\neg(\neg p \vee q)$
"EQUIV": $p \equiv q$ as $(p \longrightarrow q) \wedge (q \longrightarrow p)$

Negation $\neg$ is defined in the obvious way ($\neg(0) = 1, \neg(1) = 0$ and $\neg(\frac{1}{2}) = \frac{1}{2}$ and we will discuss definition of implication.

Implication is defined as in classical case except for $\frac{1}{2} \longrightarrow \frac{1}{2}$ that is defined as true (1) even thou $\neg\frac{1}{2} \vee \frac{1}{2} = \frac{1}{2} \vee \frac{1}{2} = \frac{1}{2}$

Lukasiewicz three valued system

P	¬ p
1	0
1/2	1/2
0	1

→	1	1/2	0
1	1	1/2	0
1/2	1	1	1/2
0	1	1	1

∧	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

Lukasiewicz three-valued system

Why Lukasiewicz did not define implication as in classical logic?

About the fight for the third value and the defense of vi-valued logic at Lukasiewicz time and in the present moment.

Can $p \to p$ be not true?

Another interpretation: Partial classical logic.

Modal operators in Lukasiewicz three-valued system

Lukasiewicz defined the following modal operators according to his intuition about the third value,

Possibility: $\Diamond p$ is true for all values of p except for false where modal operator is false. Can be defined as $\Diamond p = \neg p \rightarrow p$

Necessity: $\Box p$ is true only when p is also true and false in other cases. Can be defined as $\Box p = \neg \Diamond \neg p$

Bochvar system (1939)

Bochvar worked in decidability and found "undecidable" problems.

He proposed a three valued system where the third value means undecidable.

In that system "AND" operator changes: "undecidable AND ..... = undecidable"

The third value can also be interpreted as "paradoxical" or "meaningless".

Bochvar system (1939)

Negation is defined as in the Lukasiewicz system

"OR" is defined by duality of "AND" ($p \vee q$ as $\neg(\neg p \wedge \neg q)$)

"IMPLY" is defined as in classical logic ($p \rightarrow q$ as $\neg(p \wedge \neg q)$)

"EQUIV" like in Lukasiewicz or in classical logic
($p \leftrightarrow q$ as $(p \rightarrow q) \wedge (q \rightarrow p)$)

Bochvar three valued system

P	¬ p
1	0
1/2	1/2
0	1

→	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	1/2
0	1	1/2	1

∧	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	1/2
0	0	1/2	0

Kleene system (1938)

In Kleene systems, a proposition is to bear the third truth-value E not for fact related, ontological reasons, but for knowledge related, epistemological ones; it is not to be excluded that the proposition may "in fact" be true or false, but it is *merely unknown or undeterminable* what its specific status may be.

The connectives coincide with Lukasiewicz system except for implication ($E \rightarrow E = E$) since Kleene takes classical definition of implication ($p \rightarrow q = \neg p \vee q$).

Relation with algorithms and Partial classical logic.

Kleene three valued system

P	¬ p
1	0
1/2	1/2
0	1

→	1	1/2	0
1	1	1/2	0
1/2	1	1/2	1/2
0	1	1	1

∧	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

Kleene system (1938)

Moreover Kleene also introduce another family of connectives (weak ones) characterized by the fact that the truth tables of such connectives automatically shows the output truth value E when any one of the inputs truth value is E. With this new connectives Kleene obtain the same system as Bochvar.

Kleene motivated this truth tables in terms of arithmetical propositional functions with respect to which the key issue is that of recursiveness, i.e. being able to decide the truth status of a statement by means of an effective calculating procedure. We can think here of a mechanism that is simply *incapable of processing indeterminate statements*. Here E plays the role of a factor of all-corrupting meaninglessness, exactly as in the Bochvar system (not surprising since the motivating considerations of the two approaches are quite similar.

Post system (1921)

The Post system is a n-valued system where the values are $\{1, 2, ...n\}$, where 1 means true and n means false.

Negation is defined as a cycle $(\neg k = k + 1, \neg n = 1)$

"OR" as in the previous system $(v(p \vee q) = min(v(p, v(q)))$

"AND" as "dual" of "AND" $(v(p \wedge q) = \neg(\neg p \vee \neg q) = max(v(p), v(q)))$

Implication as in classical case $(v(p \rightarrow q) = \neg v(p) \vee v(q))$

Tautologies and designated values in multi-valued systems

In classical logic: Tautologies as formulas that takes the value true.

This can be generalized to many-valued logics (this is the more used concept of tautology)

But, more in general, we can chose a set of *designated* values ($D \subseteq V$) containing true and not false (the most usual case is an interval of the truth values chain).

Soudness of inference rules depending of the designated values.

Tautologies and designated values in Lukasiewicz systems

Some formulas that are classical tautologies and not Lukasiewicz tautologies:

EXCLUDED MIDDLE LAW: $plor\neg p$

CONTRADICTION LAW: $\neg(p \wedge \neg p)$

$[(p \to q) \to p] \to p$

Soudness of MP

But if you take $\{1, E\}$ as designated values, then tautologies of Lukasiewicz coincide with classical tautologies.

Truth-functionality

In classical logic any function $f : [0,1]^n \longrightarrow [0,1]$ can be obtained as a truth function of a classical formula. This is not the case in many-valued logics.

In classical logic this could be built using the columns of the truth table and converting them in a disjunctive normal form formula. (See example)

This is not possible in general in many valued systems due to the existence of more then two truth-values.

Relation with logical circuits (Mukaidono, Kandel)

Truth functionality in Classical Logic

p	q	r	f(p,q,r)
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	F

$f:\{0,1\}^3 \rightarrow \{0,1\}$

Truth functionality in Classical Logic

p	q	r	f(p,q,r)
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	F

(p & q & r) ∨
(p & q & ¬ r) ∨
(¬ p & q & ¬ r) ∨
(¬ p & ¬ q & ¬ r)

Excercises

What are tautologies (for 1 and for $\{1, E\}$ as set of designated values) in Bochvar, Kleene and Post systems?

Give a function in three variables in Lukasiewicz three valued logic that do not correspond to a logical formula

Read and comment the introduction of "Metamathematics of Fuzzy Logic" by P.Hàjek.

Look after the concept of supervaluation and the critique for the lost of *excluded-middle and contradiction* laws in many-valued systems.

Many-valued systems: Syntax

Language:

- Denumerable set of propositional variables
- Primary connectives
- definable connectives

Define the usual well formed formulas.

Logical system

Can be defined by:

- Axioms and inference rules (Hilbert style axiomatization)
- Gentzen systems
- Consequence operators

Many-valued systems: Semantics

General algebraic semantics

An algebra define a calculus corresponding to some logical system if it has as much operations as logical connectives and with the same arity.
Example.- The interpretation of "AND" must be a binary operation.

[0, 1]-based Semantics:

Algebraic structures over [0,1] corresponding to a logical system defines a calculus.

Syntax, semantic, completeness, decidability, computability.

Completeness properties

Let L a logical system and **A** a family of algebras of the corresponding type. Then,

L is complete with respect to **A** iff
Theorems of L = Tautologies of **A**

L is finite strong complete with respect to **A** if for all finite subset of formulas T,
$T \vdash \varphi$ if and only if $T \models_{\mathbf{A}} \varphi$

L is strong complete with respect to **A** if for all subset of formulas T
$T \vdash \varphi$ if and only if $T \models_{\mathbf{A}} \varphi$

Decidability and Computability.

Extension of Lukasiewicz calculus over [0,1]

The three valued calculus can be presented in a functional way:

NEGATION : $v(\neg p) 1 - v(p)$

AND: $v(p \wedge q) = min(v(p), v(q))$

OR: $v(p \vee q) = max(v(p), v(q))$

IMPLICATION:

$$v(p \to q) = \begin{cases} 1, & \text{if } x \leq y \\ 1 - v(p) + v(q), & \text{otherwise} \end{cases}$$

And thus extended to a system with any number of truth values closed by $+$ and 1-Id.

Extension of Lukasiewicz calculus over [0,1]

The subsets of $[0,1]$ closed by addition and 1-Id are the sets:

- The set $\{o, \frac{1}{n}, \frac{2}{n}, ... \frac{n-1}{n}, 1\}$ for all $n \in N$
- The rationals in [0,1]
- All elements of the interval [0,1].

Gödel-Dummet calculus over [0,1]

On [0,1] we can define the following calculus:

NEGATION :

$$v(\neg p) = \begin{cases} 0, & \text{if } x \geq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

AND: $v(p \wedge q) = min(v(p), v(q))$

OR: $v(p \vee q) = max(v(p), v(q))$

IMPLICATION:

$$v(p \rightarrow q) = \begin{cases} 1, & \text{if } x \leq y \\ v(q), & \text{otherwise} \end{cases}$$

And thus extended to a system with any number of truth values

Studied system before fuzzy sets where defined

Lukasiewicz systems

The calculus over L_n forms a strict chain and the calculus over rational numbers and reals over [0,1] coincide.

All are axiomatizable by a Hilbert system with Modus Ponens as inference rule (Roser-Roser, 1958, Chang 1959) (Cignoli, d'Ottsviano,Mundicci,)

Gödel-Dummett systems

The calculus over G_n forms a strict chain and the calculus over any infinite subset of [0,1] coincide.

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Predicates Calculus

As in classical logic we can study the predicate calculus of many-valued systems.

In Hjek's book Predicate Calculus obtained adding "for all" and "Exists" are studied.

Problem with quantifiers in many-valued ("fuzzy") logics.

Vague Predicates and Fuzzy Sets

In case of vaguue predicates It seems more clear that the universe is divided in at least three sets.
The prototypes (clearly satisfying the predicate, has value 1 for that property)
The anti-prototypes (clearly not satisfying the predicate, has value 0 for that property)
The intermediate (partially satisfying the property, has value in (0,1)).

If for each element we know the value, we have a function from the elements of the universe in [0,1].

This is the concept of Fuzzy set as a membership function.

T-norm based fuzzy logics: general framework

Language:

- primary connectives: $\&$, $(\wedge,)$ $\rightarrow$, $\overline{0}$
- definable connectives: $\neg$, $\vee$, $(\wedge,)$ $\leftrightarrow$

[0, 1]-based Semantics:

- $e(\varphi \wedge \psi) = \min(e(\varphi), e(\psi))$
- $e(\varphi \& \psi) = e(\varphi) * e(\psi)$, with $*$ being a left cont. t-norm
- $e(\varphi \rightarrow \psi) = e(\varphi) \Rightarrow e(\psi)$, where $\Rightarrow$ is the residuum of $*$

- a t-norm $*$ has residuum iff it is left-continuous

- each (left-continuous) t-norm $*$ determines a calculus

Monoidal t-norm based logic

MTL logic was defined in [GE, 2001] as a propositional logic in the language $\mathcal{L} = \{\&, \rightarrow, \wedge, \overline{0}\}$ defined by the following axioms:

(A1)	$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(A2)	$(\varphi \& \psi) \rightarrow \varphi$
(A3)	$(\varphi \& \psi) \rightarrow (\psi \& \varphi)$
(A4)	$(\varphi \wedge \psi) \rightarrow \varphi$
(A5)	$(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$
(A6)	$(\varphi \& (\varphi \rightarrow \psi)) \rightarrow (\varphi \wedge \psi)$
(A7a)	$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \& \psi) \rightarrow \chi)$
(A7b)	$((\varphi \& \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
(A8)	$((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
(A9)	$\overline{0} \rightarrow \varphi$

The rule of inference of MTL is *modus ponens*.

Monoidal t-norm based logic - II

Definable connectives are:
$\overline{1}$ is $\varphi \to \varphi$, $\quad \neg\varphi$ is $\varphi \to \overline{0}$,
$\varphi \vee \psi$ is $((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$.

Algebraic semantics: given by the variety of MTL-algebras, i.e. bounded commutative integral residuated lattices satisfying pre-linearity.

[JM, 2001]: (Strong) standard completeness with respect to the family MTL-chains over the real unit interval [0,1], which are defined by left continuous t-norms and their residua.

MTL is the weakest t-norm based fuzzy logic

Some well-known Extensions of MTL

Basic Fuzzy logic [Hájek, 98] (* continuous t-norms)

BL = MTL + $\varphi \wedge \psi \to (\varphi \& (\varphi \to \psi))$

Łukasiewicz logic ($x * y = \max(0, x + y - 1)$)

Ł = BL + $\neg\neg\varphi \to \varphi$

SBL = BL + $\neg(\neg\varphi \wedge \varphi)$

Gödel-Dummet logic ($x * y = \min(x, y)$)

G = (MTL, BL, SBL) + $\varphi \to \varphi \& \varphi$

Product logic ($x * y = x \cdot y$)

Π = SBL + $\neg\neg\varphi \to ((\psi \& \varphi \to \chi \& \varphi) \to (\psi \to \chi))$

Outline

- Introduction

- Truth-degrees in the syntax: on Pavelka's approach

- Algebrization of rational expansions of MTL and their axiomatic extensions

  - The case of Gödel and Nilpotent Minimum logics

- Other semantics for truth-degrees

Introducing truth-values into the language

Fuzzy logics are logics of vague, gradual properties.

T-norm based fuzzy logics are logics of comparative truth:

$$T \vdash \varphi \rightarrow \psi \text{ iff } truth(\varphi) \leq truth(\psi)$$

But to reason explicitly with partial degrees of truth, it is natural to *syntactically* introduce truth values in the language [Pavelka, 79],[Novák et al, 99],[Gerla, 01].

$$T \vdash \overline{r} \rightarrow \psi \text{ iff } r \leq truth(\psi)$$

Interest for knowledge representation purposes, e.g. in fuzzy description logics, fuzzy logic programming systems (weighted Horn-rules $(H \leftarrow B, r)$), fuzzy IF-THEN rules, . . .

Pavelka Logic

Pavelka introduced in [Pavelka, 79] a propositional many-valued logical system, denoted here PL, over Lukasiewicz logic by adding into the language as many truth constants as truth values (a truth constant $\overline{r}$ for each real $r \in [0,1]$) and some additional axioms.

This logic is not (strongly) complete w.r.t. the standard Lukasiewicz algebra $[0,1]_L$ (interpreting each $\overline{r}$ as the real r).

$$\{\overline{r} \to \varphi \mid r < \alpha\} \not\vdash \overline{\alpha} \to \varphi$$

But, Pavelka proved that PL is indeed complete in a weaker sense.

Pavelka style completeness

Truth degree of φ in T:

$$\| \varphi \|_T = \inf\{e(\varphi) \mid e \text{ evaluation model of } T\}$$

Provability degree of φ in T:

$$| \varphi |_T = \sup\{r \mid T \vdash_{PL} \overline{r} \to \varphi\}.$$

Pavelka-style completeness: $\| \varphi \|_T = | \varphi |_T$

Moreover, if the language is extended with any further connective Pavelka-style completeness is preserved iff the corresponding truth-function on the real unit interval is a continuous (real) function.

Rational Pavelka Logic (Hájek)

PL was significantly simplified in [Hájek, 98]. The language is expanded by countably-many truth-constants, one $\overline{r}$ for each rational $r \in [0,1]$. The logic is an extension of Lukasiewicz logic by the so-called book-keeping axioms:

$$\overline{r} \& \overline{s} \leftrightarrow \overline{r * s}$$
$$\overline{r} \to \overline{s} \leftrightarrow \overline{r \Rightarrow s}$$

where $*$ and $\Rightarrow$ denote the Lukasiewicz truth-functions.

This system, called Rational Pavelka Logic (RPL), satisfies the same Pavelka-style completeness and it is strongly complete for finite theories. Also corresponding predicate calculus RPL$\forall$ is defined.

Other rational expansions

Similar *rational* extensions for other popular residuated fuzzy logics can be defined, but Pavelka-style completeness strongly relies on the continuity of Lukasiewicz truth-functions in $[0, 1]$.

Lukasiewicz logic is the only t-norm based fuzzy logic with continuous truth-functions $\Longrightarrow$ we cannot expect fully analogous logics with rational truth-constants.

Product logic connectives are all continuous except for product logic implication at (0, 0).

$$x \Rightarrow_{\Pi} y = \begin{cases} 1, & \text{if } x \leq y \\ y/x, & \text{otherwise} \end{cases}$$

In rational Product logic RΠ, Pavelka-style completeness can be *formally* recovered if an infinitary inference rule is added to the logic:

$$\frac{\{\varphi \rightarrow \overline{r} \mid r \text{ rational} \in (0,1]\}}{\varphi \rightarrow \overline{0}}$$

- [EGHN, 98] for RΠ and $R\Pi_{\sim}$ (Product logic with an involutive negation)

- also in [EGM, 00] for the infinitary rational expansion RŁΠ of Łukasiewicz + Product logic ŁΠ.

$\Longrightarrow$ different techniques with other *more discontinuous* logics . . .

- [Hájek, 98]: an extension of G_{Δ} with finitely-many truth-constants, [EGHN,98]: rational G∼

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  - The case of Gödel and Nilpotent Minimum logics

  of Weak Nilpotent

- Other semantics for truth-degrees

Algebraization of Pavelka-style rational extensions

Let L be an axiomatic extension of MTL and let $*$ be a (left) continuous t-norm such that L is complete wrt $[0,1]_*$, and $*$ is closed over $[0,1] \cap \mathbb{Q}$.

Propositional logic $RL(*)$:

language: $\mathcal{L}^{\mathcal{R}} = \mathcal{L} \cup \{\overline{r} : r \in \mathbb{Q} \cap (0,1]\}$

axioms: those of L plus 'book-keeping axioms':

$$\overline{r} \& \overline{s} \leftrightarrow \overline{r * s}$$
$$\overline{r} \wedge \overline{s} \leftrightarrow \overline{\min(r,s)}$$
$$(\overline{r} \rightarrow \overline{s}) \leftrightarrow \overline{r \Rightarrow s}$$

proof: $\Gamma \vdash_{RL(*)} \varphi$

RL(∗)-algebras: $\mathcal{A} = \langle A, \&, \rightarrow, \wedge, \vee, \{\overline{r}^{\mathcal{A}} : r \in \mathbb{Q} \cap [0,1]\}\rangle$

1. $\langle A, \&, \rightarrow, \wedge, \vee, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\rangle$ is an L-algebra,

2. for every $r, s \in \mathbb{Q} \cap [0,1]$:

$$\overline{r}^{\mathcal{A}} \& \overline{s}^{\mathcal{A}} = \overline{r * s}^{\mathcal{A}}$$
$$\overline{r}^{\mathcal{A}} \wedge \overline{s}^{\mathcal{A}} = \overline{\min(r,s)}^{\mathcal{A}}$$
$$\overline{r}^{\mathcal{A}} \rightarrow \overline{s}^{\mathcal{A}} = \overline{r \Rightarrow s}^{\mathcal{A}}$$

standard RL(*)-algebra: $A = [0,1]$, $\& = *$, and $\overline{r}^{\mathcal{A}} = r$ for all rational r

evaluations: $e : \mathcal{L}^{\mathcal{R}} \rightarrow \mathcal{A}$ such that $e(\overline{r}) = \overline{r}^{\mathcal{A}}$

logical consequence: $\Gamma \models_{\mathcal{A}} \varphi$, $\Gamma \models_{RL(*)} \varphi$

Some general results

- $RL(*)$ is an algebraizable logic whose equivalent algebraic semantics is the variety of $RL(*)$-algebras.

- $RL(*)$ is complete with respect the class of $RL(*)$-chains.

- Conservativeness of $RL(*)$ wrt to L.

- If $*$ and $*'$ are isomorphic t-norms, then $RL(*)$ and $RL(*')$ are mutually translatable.

Question: what about standard completeness . . .

Weak nilpotent t-norms

They are of the form:

$$x * y = \begin{cases} \min(x, y), & \text{if } x \geq n(y) \\ 0, & \text{otherwise} \end{cases}$$

where $n : [0, 1] \rightarrow [0, 1]$ is a weak negation function, that is, a non increasing function with $n(1) = 0$ and $n(n(x)) \geq x$.

Two important particular cases:

Minimum (negation is Gödel negation)
Nilpotent minimum (negation is an involution)

Their corresponding logics

Weak nilpotent minimum: WNM = MTL +

$$(\varphi \& \psi \to \overline{0}) \vee (\varphi \wedge \psi \to \varphi \& \psi)$$

Gödel Logic: G = WNM (MTL, BL) + $\varphi \to \varphi \& \varphi$

Nilpotent Minimum: NM = WNM + $\neg\neg\varphi \to \varphi$

- G and NM have standard completeness w.r.t. a single t-norm. WNM is standard complete w.r.t. a family of t-norms.
- G satisfies the usual deduction theorem. WNM (and NM) satisfies a weaker form:
  $$\Gamma, \psi \vdash_{WNM} \varphi \text{ iff } \Gamma \vdash_{WNM} (\psi \& \psi) \to \varphi$$

Rational Gödel Logic (RG)

RG logic $= RG(\text{min})$

• RG-chains are G-chains where the rational truth constants are located in the same strict order than the rationals except for an order filter that colapses to the maximum 1.

• Weak standard completeness: $\vdash_{RG} \varphi$ if and only if $\models_{[0,1]_{RG}} \varphi$

• RG does not have strong standard completeness, even for finite theories. (neither Palveka-style completeness)

Counterexample: let $T = \{\overline{r} \vee p\}$ with $r < 1$, then

$$T \models_{[0,1]_{RG}} p \quad \text{but} \quad T \not\vdash_{RG} p.$$

Restricted strong standard completeness for RG

Restricted set of formulas: formulas of the type $\overline{r} \rightarrow \varphi$, where φ does not contain intermediate truth-constants (it is a G-formula).

$\overline{r} \rightarrow \varphi$ :: truth-value(φ) $\geq r$ alternative notation (φ, r)

Finite Strong Completeness:
Let $\Gamma = \{(\psi_i, r_i) \mid i = 1, 2, \ldots, n\}$. Then:

$$\Gamma \vdash_{RG} (\varphi, s) \text{ iff } \Gamma \models_{[0,1]_{RG}} (\varphi, s)$$

Rational Nilpotent minimum Logic

RNM logic $= RNM(*_{NM})$

• RNM-chains are *rotations* of RG-chains

(actually, NM-chain = rotation of a Gödel hoop [Jenei], [Wronski])

• Weak standard completeness: $\vdash_{RNM} \varphi$ if and only if $\models_{[0,1]_{RNM}} \varphi$

• RNM is (finitely) strongly standard complete if we restrict ourselves to formulas of the type $\overline{r} \to \varphi$ with $r > \frac{1}{2}$

Completeness: Let $\Gamma = \{(\psi_i, r_i) \mid i = 1, 2, \ldots, n\}$ with $r_i > 1/2$. Then:

$$\Gamma \vdash_{RNM} (\varphi, s) \text{ iff } \Gamma \models_{[0,1]_{RNM}} (\varphi, s)$$

Other rational expansions of WNM logics

No general result, but similar results for three new extensions of WNM:

- WNM_* = WNM + $(\varphi \to \varphi^2) \to ((\varphi \wedge \psi) \to (\varphi \& \psi))$
- $WNM_\star$ = WNM + $(\neg\neg\varphi \to \varphi) \vee (\neg\neg\varphi \leftrightarrow \neg\varphi)$
- $WNM_\circ$ = WNM +

$$(\neg\neg\varphi \to \varphi) \vee (\neg\neg\psi \to \psi) \vee ((\neg\varphi \leftrightarrow \neg\psi) \wedge (\neg\varphi \to \varphi))$$
$$(\neg\neg p(\varphi) \to p(\varphi)) \vee ((\neg\neg p(\psi) \to p(\psi)) \to (p(\varphi) \to p(\psi)))$$

where $p(\chi)$ denotes $\chi \vee \neg\chi$.

Similar definitions for the corresponding algebras.

Corresponding logics: completeness

Standard completeness:

- WNM_* is complete wrt $[0,1]_{*_c}$ for any $0 < c < 1$
- $WNM_\star$ is complete wrt $[0,1]_{\star_c}$ for any $1/2 < c < 1$
- $WNM_\circ$ is complete wrt $[0,1]_{\circ_c}$ for any $1/2 < c < 1$

constants are located

Results for rational expansions of WNM_*, $WNM_\star$, $WNM_\circ$

Let $+$ denote any of $*, \star, \circ$

Weak standard completeness:

$$\vdash_{RWNM(+)} \varphi \text{ if and only if } \models_{[0,1]_{+c}} \varphi$$

Finite strong standard completeness if restricted to formulas of type $\overline{r} \to \varphi$ when $r > c$:

$$\Gamma \vdash_{RWNM(+)} (\varphi, s) \quad \text{iff} \quad \Gamma \models_{[0,1]_{[0,1]_{+c}}} (\varphi, s)$$

for any $\Gamma = \{(\psi_i, r_i) \mid i = 1, 2, .., n\}$ with $r_i > c$.

Open issues

- Can the results be extended to any other standard complete WNM logics ?
- Rational expansions for other left continuous t-norm logics, in particular for continuous t-norm logics
- Rational expansions of predicate fuzzy logics (first results for $\forall$G)
- Rational expansions and proof theory

Outline

- Introduction
- Truth-degrees in the syntax: on Pavelka's approach
- Algebrization of rational expansions of MTL and their axiomatic extensions
  - The case of Gödel and Nilpotent Minimum logics
- Other semantics for truth-degrees

Other (additional) meanings of truth-degrees

Rational expansions of some t-norm based logics can be used as base frameworks where to define modalities, modelling a graded intensional notion, applying to crisp (or even fuzzy) propositions.

Examples:

- being probable, likely are typical fuzzy predicates.
- Also for similar predicates (or modalities) related to other uncertainty measures (possibility/necessity, belief functions).
- truth-likeness

This process can be iterated to several levels

Example (2 levels) : reasoning about probability

φ: classical proposition
$P\varphi$: fuzzy proposition "φ is probable"

$truth - degree(P\varphi) = probability(\varphi)$

Axioms governing probability laws: extend RPL (or ŁΠ$\frac{1}{2}$) by

CP axioms for non modal φ

(FP1) $P(\varphi) \rightarrow_L (P(\varphi \rightarrow \psi) \rightarrow_L P(\psi))$
(FP2) $P(\neg\varphi) \equiv_L \neg_L P(\varphi)$
(FP3) $P(\varphi \vee \psi) \equiv_L (P(\varphi) \rightarrow_L P(\varphi \wedge \psi)) \rightarrow_L P(\psi)$.

Many-valued semantics for $P\varphi$'s, probabilistic semantics for φ's [HGE,95,00]

Example (3 levels) : reasoning about belief functions

φ: classical proposition
$\Box\varphi$: classical modal S5 proposition
$B\varphi = P\Box\varphi$: fuzzy proposition "φ is believed"

$$truth - degree(B\varphi) = probability(\Box\varphi) = belief(\varphi)$$

Axioms governing DS belief functions: extend RPL (or $\text{Ł}\Pi\frac{1}{2}$) by

S5 axioms for $\Box$-formulas
axioms (FP1), (FP2), (FP3) for P-formulas

Many-valued semantics for $P\Box\varphi$'s, probabilistic semantics for $\Box\varphi$'s, belief function semantics for φ. [GHE, 01]

The End