Similarity based reasoning and its logical characterization

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similarity

- Uncertainty: measures of uncertainty
- Vagueness: many-valued logic
- Analogy: Similarity-based reasoning

Outline

- Similarity-based entailment relations
  - Approximate entailment
  - Proximity entailement
  - Strong entailment
- Logical formalizations: Conditional and modal approaches
- Conclusions

Truthlikeness

φ_1: there are 300 steps to the top of *Pisa tower*

φ_2: there are 1000 steps to the top of *Pisa tower*

In the real world both are false (there are 296!),

but φ_1 provides a more accurate description,
it is more close to be true, than φ_2

why?

300 is more similar to 296 than 1000

Truthlikeness II

Similarity-based truthlikeness

- a more informed scenario (e.g. real world w_0 + similarity S)

$$Truth = \{\varphi \mid w_0 \models \varphi\} \quad Falsity = \{\psi \mid w_0 \models \neg\psi\}$$

$$\alpha\text{-Truthlike} = \{\psi \mid w \models \psi, S(w_0, w) \geq \alpha\}$$

- more fine-grained representation and reasoning framework: given a theory (epistemic state), one can identify not only its true, false or undecided consequences, but also which consequences are closer (more truth-like) to hold than others
- it allows to supports patterns of some forms of analogical reasoning, e.g. in approximate or case-based reasoning

Aim: to show some logical formalizations of some forms of (degree-based) similarity-based reasoning

Graded similarity (indistinguishability) relations

A $\otimes$-similarity relation on D is a mapping $S : D \times D \to [0, 1]$ usually required to satisfy dual properties of those of a (bounded) metric:

- **Reflexivity**: $S(u, u) = 1$
  Separation: $S(u, v) = 1$ only if $u = v$
- **Symmetry**: $S(u, v) = S(v, u)$
- $\otimes$-**Transitivity**: $S(u, v) \otimes S(v, w) \leq S(u, w)$

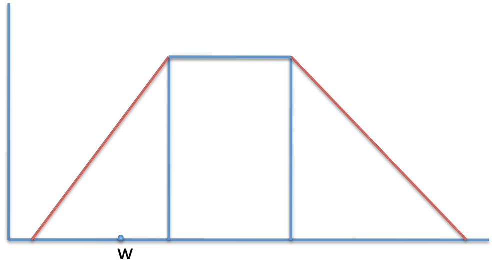
- when $x \otimes y = \max(x + y - 1, 0)$ and S separating, then $\delta = 1 - S$ is a distance

- when $x \otimes y = \min(x, y)$, then $\delta = 1 - S$ is an ultrametric

Weaker notions: closeness relations (Refl), proximity relations (Refl + Sim)

Links to similarity-based semantics for fuzzy sets
E.H. Ruspini. "On the Semantics of Fuzzy Logic," Int.J. Approximate Reasoning, 5 , 45-88, 1991.

Similarity and Fuzzy Sets



Classical set A and a similarity relation S ⟶ the fuzzy set A*

Membership of A* in w is Sup {S(w,v) / v |= A}

Truthlikeness degrees and similarity measures

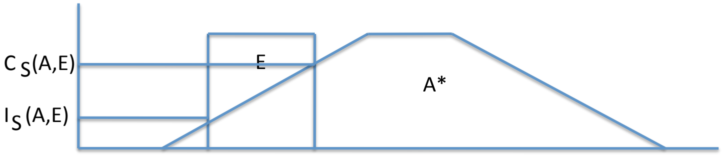
Take CPC and the set of Boolean interpretations (W)
Given $S : W \times W \to [0,1]$

Truthlikeness degree of φ at $w \in W$: $I_S(\varphi \mid w) = \sup_{w' : w' \models \varphi} S(w, w')$

(Ruspini, 91)'s definitions:

- **Implication measure**: $I_S(\varphi \mid \psi) = \inf_{w : w \models \psi} I_S(\varphi \mid w)$
- **Consistency measure**: $C_S(\varphi \mid \psi) = \sup_{w : w \models \psi} I_S(\varphi \mid w)$

Implication and consistency measures



$$I_S(A,E) = \mathrm{Inf}\,\{A^*(w) \mid w \models E\}$$

$$C_S(A,E) = \mathrm{Sup}\,\{A^*(w) \mid w \models E\}$$

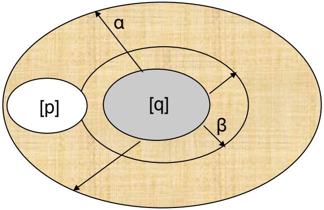
Truthlikeness degrees and similarity measures

$S : W \times W \to [0,1]$

$\Rightarrow$ spheres around the set of models of a proposition $[q]$

$$U_\alpha([q]) = \{w \in W \mid \text{ exists } w' \in [q] \text{ and } S(w', w) \geq \alpha\}$$

$$[q] = U_1([q]) \subseteq \ldots \subseteq U_\alpha([q]) \subseteq \ldots \subseteq U_0([q]) = W$$



$\alpha = I_S(q \mid p) = \sup\{\delta \mid [p] \subseteq U_\delta([q])\}$ "p α-approximately entails q"

$\beta = C_S(q \mid p) = \sup\{\delta \mid [p] \cap U_\delta([q]) \neq \emptyset\}$ "p is β-compatible with q"

Approximate and Strong entailments

Given $S : W \times W \to [0,1]$ define

Approximate entailment (DEGGP,97):

$$\varphi \models^{\alpha}_{S} \psi \quad \text{iff} \quad \text{for all } \omega,\ \omega \models \varphi \text{ implies } \exists \omega' : \omega \models \psi \text{ and } S(\omega,\omega') \geq \alpha$$
$$\text{iff} \quad [\varphi] \subseteq U_{\alpha}([\psi])$$

Strong entailment (EGRV12):

$$\varphi \approx^{\alpha}_{S} \psi \quad \text{iff} \quad \text{for all } \omega,\ \omega \models^{\alpha}_{S} \varphi \text{ implies } \omega \models \psi$$
$$\text{iff} \quad U_{\alpha}([\varphi]) \subseteq [\psi]$$

Approximate entailment: If φ then approximately ψ

Strong entailment: If approximately φ then ψ

D. Dubois, F.Esteva, P. Garcia, L. Godo H. Prade
A Logical Approach to Interpolation Based on Similarity Relations.
International Journal of Approximate Reasoning. North-Holland, Vol. 17 (1), 1-36 (1997).
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Logics for approximate and strong entailment
Fuzzy Sets and Systems,Elsewier, Vol 197, p.59-70 (2012)

I_S and Approximate entailment

Approximate entailment (DEGGP,97): Given $S : W \times W \to [0,1]$ define

$$p \models^{\alpha}_{S} q \quad \text{iff} \quad \text{for all } \omega,\ \omega \models p \text{ implies } \exists \omega' : \omega \models q \text{ and } S(\omega,\omega') \geq \alpha$$
$$\quad\quad\quad \text{iff} \quad [p] \subseteq U_{\alpha}([q])$$

Proposition: $p \models^{\alpha}_{S} q$ iff $I_S(q \mid p) \geq \alpha$

(1)	**Nestedness:**	if $p \models^{\alpha} q$ and $\beta \leq \alpha$ then $p \models^{\beta} q$;
(2)	$\otimes$-**Transitivity:**	if $p \models^{\alpha} r$ and $r \models^{\beta} q$ then $p \models^{\alpha \otimes \beta} q$;
(3)	**Reflexivity:**	$p \models^{1} p$;
(4)	**Right weakening:**	if $p \models^{\alpha} q$ and $q \models r$ then $p \models^{\alpha} r$;
(5)	**Left strengthening:**	if $p \models r$ and $r \models^{\alpha} q$ then $p \models^{\alpha} q$;
(6)	**Left OR:**	$p \vee r \models^{\alpha} q$ iff $p \models^{\alpha} q$ and $r \models^{\alpha} q$;

...

CUT does not hold: $p \models^{\alpha}_{S} r$ and $p \wedge r \models^{\beta}_{S}$ does not imply $p \models^{f(\alpha,\beta)}_{S} q$.

N_S and Strong entailment

Strong entailment (EGRV,10): Given $S : W \times W \to [0,1]$ define

$$\varphi \approx^{\alpha}_{S} \psi \quad \text{iff} \quad \text{for all } \omega,\ \omega \models^{\alpha}_{S} \varphi \text{ implies } \omega \models \psi$$
$$\quad \text{iff} \quad U_{\alpha}([\varphi]) \subseteq [\psi]$$

Proposition: Let $\alpha > 0$. Then
$$\varphi \approx^{\alpha}_{S} \psi \quad \text{iff} \quad N_S(\psi \mid \varphi) = 1 - C_S(\neg\psi \mid \varphi) > 1 - \alpha$$

Characterizing properties:

(1) **Nestedness:** if $\varphi \approx^{\alpha}_{S} \psi$ and $\beta \geq \alpha$ then $\varphi \approx^{\beta}_{S} \psi$;

(2) **Lower bound:** $\varphi \approx^{0}_{S} \psi$ iff either $\models \neg\varphi$ or $\models \psi$

(3) **Upper bound:** $\varphi \approx^{1}_{S} \psi$ iff $\varphi \models \psi$

(4) **min-Transitivity:** if $\varphi \approx^{\alpha}_{S} \psi$ and $\psi \approx^{\beta}_{S} \chi$ then $\varphi \approx^{\min(\alpha,\beta)}_{S} \chi$;

(5) **Left OR:** $\varphi \vee \chi \approx^{\alpha}_{S} \psi$ iff $\varphi \approx^{\alpha}_{S} \psi$ and $\chi \approx^{\alpha}_{S} \psi$;

(6) **Right AND:** $\chi \approx^{\alpha}_{S} \varphi \wedge \psi$ iff $\chi \approx^{\alpha}_{S} \varphi$ and $\chi \approx^{\alpha}_{S} \psi$.

(7) **Contraposition:** if $\varphi \approx^{\alpha}_{S} q$ then $\neg q \approx^{\alpha}_{S} \neg\varphi$

...

Another example

Rule R:

If x has more than 1.000 euros, then x should invest half of the capital

Facts: John has 200 euros and Mark has 990 euros

Using classical logic the application of the rule gives the same answer for both: nothing

This seems reasonable for John but not for Mark
since Mark is close to satisfy the premise

It seems reasonable that

If the premises are close to be true
the consequence also have to be close to be true

We need a more flexible application of the rule......

Yet another type: Proximity entailment

Approximate entailment: $\varphi \models^{\alpha}_{S} \psi$ holds iff $[\varphi] \subseteq U_{\alpha}([\psi])$

What if we allow a graceful progagation of this relaxation to neighborhoods of φ and ψ?

$$U_{\beta}([\varphi]) \subseteq U_{\alpha\otimes\beta}([\psi]), \text{ for any } \beta$$

Proximity entailment:

$$\begin{aligned} \varphi \models\!\!\!\equiv^{\alpha}_{S} \psi \quad & \text{iff} \quad U_{\beta}([\varphi]) \subseteq U_{\alpha\otimes\beta}([\psi]), \text{ for any } \beta \\ & \text{iff} \quad \text{for all } \omega \text{ and } \beta,\ \omega \models^{\beta} \varphi \text{ implies } \omega \models^{\alpha\otimes\beta} \psi \\ & \text{iff} \quad J_S(\psi \mid \varphi) = \inf_w I_S(\varphi \mid w) \Rightarrow_{\otimes} I_S(\varphi \mid w) \geq \alpha \end{aligned}$$

" If approximately φ then approximately* ψ "

Compare: Approximate entailment: If φ then approximately ψ
Strong entailment: If approximately φ then ψ

Yet another type: Proximity entailment

But, due to the $\otimes$-transitivity of S, $U_\beta \circ U_\alpha \subseteq U_{\beta\otimes\alpha}$, hence
$\varphi \models\!\!\!\equiv^\alpha \psi$ iff $\varphi \models^\alpha \psi$ (proximity = approximate !)

Background knowledge: relativized entailments and measures

Assume the real world ω_0 is among those satifying K:

$$\begin{array}{lll} \varphi \models^\alpha_K \psi & \text{iff} & \text{for all } \omega \in [K],\ \omega \models \varphi \text{ implies } \omega \models^\alpha \psi, \\ & \text{iff} & K \wedge \varphi \models^\alpha \psi \end{array}$$

$$\begin{array}{lll} \varphi \models\!\!\!\equiv^\alpha_K \psi & \text{iff} & \text{for all } \omega \in [K] \text{ and } \beta,\ \omega \models^\beta \varphi \text{ implies } \omega \models^{\alpha\otimes\beta} \psi \\ & \text{iff} & J_K(\psi \mid \varphi) = \inf_{\omega\in[K]} I_S(\varphi \mid \omega) \Rightarrow_\otimes I_S(\varphi \mid \omega) \geq \alpha \end{array}$$

$\varphi \models\!\!\!\equiv^\alpha_K \psi$: "In the context of K, if approximately φ then approximately ψ"

Extrapolation of the "If φ then ψ" relation to the vincinity of both φ and ψ.

$\varphi \models\!\!\!\equiv^\alpha_K \psi$ and $\varphi \models^\alpha_K \psi$ are no longer equivalent !

Vagueness and Approximate consequences: "Heap" example

$Var = \{h_0, h_1, \ldots, h_{1000}\}$

h_n: n grains of sand form a heap

$K = \{h_{1000}\} \cup \{h_n \rightarrow h_m \mid n \leq m\}$

Sorites paradox:

$\{h_n \rightarrow h_{n-1} : n \leq 1000\} \cup K \models h_0$

Vagueness and Approximate consequences: "Heap" example

$Var = \{h_0, h_1, \ldots, h_{1000}\}$

h_n: n grains of sand form a heap

$K = \{h_{1000}\} \cup \{h_n \to h_m \mid n \leq m\}$

$\Omega_K = \{\omega_0, \ldots, \omega_{1000}\}$, where $\omega_n(h_m) = 1$ if $n \leq m$, $\omega_n(h_m) = 0$ other.

$S : \Omega_K \times \Omega_K \to [0, 1]$ is defined as

$$S(\omega_n, \omega_m) = 1 - \frac{|n - m|}{1000}$$

S is a $\otimes_{Ł}$-similarity ($1 - S$ is a distance)

Then we have:

$$h_n \models^{0.999}_{S,K} h_{n-1}$$

and

$$h_{1000} \models^{0.001}_{S,K} h_1$$

Combining Proximity and Approximate consequences:

Suppose we have a classical rule and a similarity relation S such that

$$\gamma \models^{\beta}_{S,K} \psi$$

and a premise φ such that

$$\varphi \models^{\alpha}_{S} \gamma$$

Then, we have the following pattern of similarity-based reasoning:

$$\frac{\varphi \models^{\alpha}_{S} \gamma \qquad \gamma \models^{\beta}_{S,K} \psi}{\varphi \models^{\alpha\otimes\beta}_{S,K} \psi}$$

Proximity and Approximate consequences: **case-based reasoning**

CBR: problem solving method in AI based on the principle that

"Similar problems have similar solutions"

Given a base of already solved problems (cases) and a new problem, the CBR cycle is:

1. RETRIEVE the most similar case(s)
2. REUSE the information and knowledge in that case(s) to solve the problem
3. REVISE the proposed solution
4. RETAIN the parts of this experience likely to be useful for future problem solving

Objects: described by a set $\mathcal{A}$ of attributes $\mathbf{d} = (a^1, \dots a^r)$
Classes: $\mathcal{CL} = \{class^1, \dots class^m\}$
$BC = \{(\mathbf{d_i}, class_i) \mid i = 1, \dots n\}$: case-base of already classified objects
$\mathbf{d}^*$: new problem

K = "The more similar is $\mathbf{d}^*$ to $\mathbf{d_i}$,
the more plausible $class_i$ is the class for d^*"

Given $\otimes$-similarities S_1 on $\mathcal{A}^n$ and S_2 on $\mathcal{CL}$ and let $S = S_1 \times S_2$. Then

$$\frac{\mathbf{d}^* \models^{\alpha}_{S} \mathbf{d_i} \qquad \mathbf{d_i} \models^{\beta}_{S,K} class_i}{\mathbf{d}^* \models^{\alpha\otimes\beta}_{S,K} class_i}$$

Assign to $\mathbf{d}^*$ the class which is an approximate consequence with highest degree.

- Classification of Schistosomiasis Prevalence Using Fuzzy CBR (IWANN 09) -

Outline

- Similarity-based entailment relations and its modal representation
  - Approximate entailment
  - Proximity entailement
  - Strong entailment
  - Logical formalizations: Conditional approaches
- Conclusions

Logics of approximate and proximity entailments

Aim: encode the graded entailments $\varphi \models^\alpha_S \psi$, $\varphi \approx^\alpha_S \psi$ and $\varphi \equiv^\alpha_S \psi$ at the object level as a conditional-like formula

Language:

- propositional formulas are conditional formulas
- if φ, ψ propositional formulas and $\alpha \in C$, then

$$\varphi >_\alpha \psi \qquad \varphi \succ_\alpha \psi \qquad \varphi \gg_\alpha \psi$$

are conditional formulas; no nested conditional formulas!

Semantics: $M = (W, S, e)$

$(M, w) \models \varphi >_\alpha \psi$ if $I_S(\psi \mid \varphi) \geq \alpha$ (independent of w)

$(M, w) \models \varphi \succ_\alpha \psi$ if $N_S(\psi \mid \varphi) \geq \alpha$

$(M, w) \models \varphi \gg_\alpha \psi$ if $I_S(\varphi \mid w) \Rightarrow_\otimes I_S(\psi \mid w) \geq \alpha$,

LAE: a logic of approximate entailment

Axioms:

(CPC)	tautologies of CPC
(N)	$\varphi >_\alpha \psi \to \varphi >_\beta \psi$ if $\beta \leq \alpha$
(CS)	$\varphi >_1 \psi \to (\varphi \to \psi)$
(EX)	$\varphi >_0 \psi$
(B)	$\chi >_\alpha \chi' \to \chi' >_\alpha \chi$, if χ and χ' are m.e.c.'s
(4)	$(\varphi >_\alpha \psi) \wedge (\psi >_\beta r) \to \varphi >_{\alpha \otimes \beta} r$
(LO)	$(\varphi \vee \psi >_\alpha r) \leftrightarrow (\varphi >_\alpha \chi) \wedge (\psi >_\alpha \chi)$
(RO)	$(\chi >_\alpha \varphi \vee \psi) \leftrightarrow (\chi >_\alpha \varphi) \vee (\chi >_\alpha \psi)$, if χ is a m.e.c.

Rule:

(RK) From $\varphi \to \psi$ infer $\varphi >_1 \psi$

Completeness results (Rodriguez, 02): T finite, $T \vdash_{LAE} \Phi$ iff $T \models_{LAE} \Phi$

LSE: a logic of strong entailment

Axioms:

$(A1)$	uniform substitutions of CPC tautologies
$(A2)$	$\varphi \succ_1 \psi$, where $\varphi \to \psi$ is a tautology of CPL
$(A3)$	$\bot \succ_0 \varphi, \quad \varphi \succ_0 \top$
$(A4)$	$(\varphi \succ_\alpha \psi) \wedge (\varphi \succ_\alpha \chi) \to (\varphi \succ_\alpha \psi \wedge \chi)$
$(A5)$	$(\varphi \succ_\alpha \chi) \wedge (\psi \succ_\alpha \chi) \to (\varphi \vee \psi \succ_\alpha \chi)$
$(A6)$	$(\varphi \succ_\alpha \psi) \to (\neg\psi \succ_\alpha \neg\varphi)$
$(A7)$	$(\varphi \succ_\beta \psi) \wedge (\psi \succ_\alpha \chi) \to (\varphi \succ_{\min(\beta,\alpha)} \chi)$
$(A8)$	$(\varphi \succ_\beta \psi) \to (\varphi \succ_\alpha \psi)$, where $\alpha \geq \beta$.
$A9$	$(\varphi \succ_0 \psi) \to (\varphi \succ_1 \bot) \vee (\top \succ_1 \psi)$

Rule: (MP) $$\frac{\Phi \quad \Phi \to \Psi}{\Psi}$$

Completeness (EGRV, 10): if $\mathcal{T}$ finite, $\mathcal{T} \vdash_{LSE} \Phi$ iff $\mathcal{T} \models_{LSE} \Phi$

LPE: logic of proximity entailment

Axioms:

CPC:	tautologies of CPC
N:	$\varphi \gg_\alpha \psi \to \varphi \gg_\beta \psi$ if $\beta \leq \alpha$
CS:	$\varphi \gg_1 \psi \to (\varphi \to \psi)$
EX:	$\varphi \gg_0 \psi$
4:	$(\varphi \gg_\alpha \psi) \wedge (\psi \gg_\beta \chi) \to \varphi \gg_{\alpha \otimes \beta} \chi$
LO:	$(\varphi \vee \psi \gg_\alpha \chi) \leftrightarrow (\varphi \gg_\alpha \chi) \wedge (\psi \gg_\alpha \psi)$
RO:	$(\chi \gg_\alpha \varphi \vee \psi) \leftrightarrow (\chi \gg_\alpha \varphi) \vee (\chi \gg_\alpha \psi)$

Rules:

From $\varphi \to \psi$ infer $\varphi \gg_1 \psi$

Completeness (Rodriguez, 2002): if T finite, $T \vdash_{LPE} \Phi$ iff $T \models_{LPE} \Phi$

To conclude

- We have presented three fields that are still under research
- Measures of uncertainty and Fuzzy Logic (Beginning in 1995 paper and extended to uncertainty measures over many-valued residuated logics)
- Similarity-based reasoning to deal with truthlikeness
- Fuzzy Description Languages and its applications
- All demanding a better understanding of modal operators of different types over many-valued logic

THANKS!