From classical to many-valued logics and reasoning with partial truth

Francesc Esteva
IIIA-CSIC, Spain

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Two non classical reasoning: dealing with vagueness and analogy

We deal with two type of non-classical reasoning:

Vague predicates via fuzzy sets

Analogical reasoning via similarity

Outline

• Vague predicates and fuzzy sets

• What does Fuzzy Logic means? Narrow and wide sense.

• Fuzzy logics in narrow sense, as multivalued residuated logics with t-norm semantics.

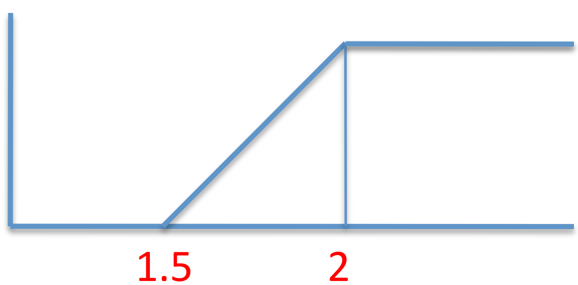
• Some logical treatment of topics of Zadeh's Fuzzy agenda

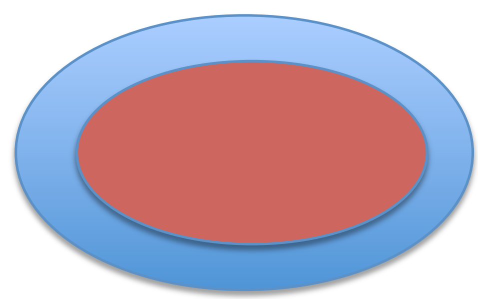
• Partial truth and Fuzzy logics with truth constants

Vagueness and Fuzzy Logic

Vagueness: Imprecission when limits are not clearly stated

Predicate high divide the univers in three subsets (depending on the context)





Gradual properties
Preference relations
Prototypes and similarities

D. Dubois, F. Esteva, L.Godo, H.Prade (2005); An Information-based discussion of vagueness, in HANDBOOK OF CATEGORIZATION IN COGNITIVE SCIENCE (Henri Cohen, Claire Lefebvre ed.), Part 8.- Machine Category learning (2005) pp. 892-913

FUZZY Sets

Fuzzy Sets are quantitative representation of vague predicates.

They are:
fuzzy subsets of a universe U (representation of context)
A fuzzy subset A is given by a membership function

$$\mu_A : U \to [0, 1]$$

Operations on Fuzzy Sets

Fuzzy subsets on a universe U are

Fuzzy subset A is defined by its membership function

$$\mu_A : U \longrightarrow [0,1]$$

Operations are defined pointwise through a function on $[0,1]$

$$\mu_{A \circ B} = T_\circ(\mu_A, \mu_B)$$

being $T_\circ$ is a binary operation on $[0,1]$ corresponding to $\circ$.

Modelización lógica

		Knowledge: Precise	Knowledge: Imprecise	Knowledge: Vague
PREDICATES	Crisp	Classical Logic	Partial classical logic	Possibility Logic
	Vague	Many-valued logic	Partial many-valued logic	Fuzzy logic (Fuzzy truth values)

FUZZY LOGIC IN THE 70-ies AND 80-ies

Fuzzy Logic was mainly devoted to operations on [0,1] corresponding to logical connectives-truth functions:

Use of t-norms to model "and" and of t-conorms to model "or"
Characterization properties for conjunction, disjunction and negation functions and their generators.
R-implication: $x \to y = \max\{z \mid z * x \leq y\}$
S-implication: $x \to y = n(x) \vee y$
Modus ponens generation functions.

FUZZY LOGIC IN THE 70-ies AND 80-ies

Algebras defined over [0,1] by these truth-functions.
Example: De Morgan triples.

Properties combining these truth-functions.

Example: (Alsina, 1988)
Given a continuous t-norm $*$, a continuous t-conorm $\oplus$ and an involutive negation N, the general solution of the functional equation

$$(x * y) \oplus (x * N(y)) = x$$

is (up to isomorphisms) $* = Product$, $\oplus =$ Lukasiewicz t-norm and N be the Lukasiewicz negation function ($N(x) = 1 - x$).

About t-norms I

A t-norm is an operation $* : [0,1] \times [0,1] \to [0,1]$ such that it satisfy:

Associativity: $x * (y * z) = (x * y) * z$
Commutativity: $x * y = y * x$
Monotonicity: If $x \leq y$, then $x * z \leq y * z$
Boundary properties: $x * 0 = 0$ and $x * 1 = x$

t-norm like (monoid): Operation over a bounded linearly ordered set satisfying the properties of a t-norm.

About t-norms II

Basic examples and Properties:

Discrete: If $x, y \neq 1$, then $x *_d y = 0$
Minimum: $x *_{min} y = \min(x, y)$
Product: $x *_\pi y = x \times y$
Lukasiewicz; $xast_L y = \max(0, x + y - 1)$

For all t-norm $*$, then $x *_d y \leq x * y \leq \min(x, y)$
For all continuous t-norm $x *_L y \leq x * y \leq \min(x, y)$

Negation functions on [0,1]

Negation functions $N : [0,1] \to [0,1]$ have to be:
Order-reversing: If $x \leq y$, then $N(x) \leq N(y)$
and $N(0) = 1, N(1) = 0$.

And may be:
Strong negation (Involutive): $NN(x)) = x$
Weak negation : $NN(x)) \geq x$
Dualy-weak negation : $NN(x)) \leq x$

Strong negation are deformations of the usual one ($N(x) = 1 - x$)
All strong negation are of the form $N(x) = f^{-1}(1 - f(x))$
being f an homeomorfisme of [0,1].

Implication functions

Its seems reasonable that a implication function have to be:
1) Non-increasing in the first variable and non-decreasing in the second variable.
2) Extension of the classical implication over $\{0, 1\}$.

Two are the main Implication functions on [0,1] used in Fuzzy setting (generalization of the classical):
S-implications. $x \rightarrow_S y = \neg x \vee y$)
R-implications ($x \rightarrow_S y = \max\{z \in [0, 1] \mid x * z \leq y\}$)

Why do you define them?

Implication and Modus Ponens

Given a t-norm $*$ it seems interesting to find a implication compatible with the following Modus Ponens rule (MP):
If $x \geq a$ and $x \to y \geq b$, then $y \geq a * b$
This can be reformulated as the following constraint:
If $z \leq x \to y$, then $x * z \leq y$

Take the maximal value of implication compatible with MP, i.e. satisfying:
$z \leq x \to y$ if and only if $x * z \leq y$

We obtain R-Implication ($x \to_R y = \max\{z \in [0, 1] \mid x * z \leq y\}$)

Properties of R-implications

If a t-norm $*$ has residuum, then it satisfy:

1) It is non-increasing in the first variable and non-decreasing in the second variable, i.e.
if $x \leq y$, then $x \to z \geq y \to z$ and $z \to x \leq z \to y$
2) $x \to x = 1$
3) $x \to y = 1$ if and only if $x \leq y$
4) $x \to y \geq y$
5) $x * (x \to y) \leq x \wedge y$

Outline

- Vague predicates and fuzzy sets
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- Partial truth and Fuzzy logics with truth constants

FUZZY LOGIC: NARROW AND WIDE SENSE (Lotfi Zadeh)

The term Fuzzy Logic has two different meanings: wide and narrow. In a narrow sense it is a logical system which aims a formalization of approximate reasoning. In this sense it is an extension of multi-valued logic.

However the agenda of Fuzzy logic is quite different from that of traditional multi-valued logic. Such key concepts in FL as the concept of linguistic variable, fuzzy if-then rule, fuzzy quantification and defuzzification, truth qualification, the extension principle, the compositional rule of inference and interpolative reasoning, among others, are not adressed in traditional systems. In its wide sense, FL, is fuzzily synonimous with the fuzzy set theory of classes of unsharp boundaries. FST is much broader than FL and includes the later as one of its branches.

FUZZY LOGIC IN NARROW SENSE (Petr Hájek)

Formal calculi of many-valued logic are the kernel of Fuzzy logic in narrow sense.

He undertakes the task of explaining Zadeh's concepts by means of these calculi.

Methamatematics of Fuzzy Logic, Kluwer, 1998

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RESIDUATED (FUZZY) MULTIPLE-VALUED LOGICS

Language: w.f.f are built from a countable set of propositional variables by means of connectives &, $\rightarrow$, $\neg$, $\wedge$, $\vee$, and constants $0, 1$.

Interpretations: Variables as elements of $[0, 1]$ and connectives as, a t-norm $*$, its residuum $\Rightarrow_*$, $n_*(x) = x \Rightarrow_* 0$, min, max, 0 and 1 respectively.

$x \Rightarrow_* y = \max\{z \mid x * z \leq y\}$

T-NORMS AND RESIDUATION

A t-norm $*$ has residuum $\Rightarrow_*$ iff it is left-continuous.

Order-definability: $x \leq y$ if and only if $x \Rightarrow_* y = 1$

Transitivity: $(x \Rightarrow_* y) * (y \Rightarrow_* z) \leq x \Rightarrow_* z$

Prelinearity: $\max((x \Rightarrow_* y), (y \Rightarrow_* x)) = 1$

max-def.: $\max(x, y) = \min(((x \Rightarrow_* y) \Rightarrow_* y), ((y \Rightarrow_* x) \Rightarrow_* x)))$

No-divisibility: for all $x \leq z$ there exists t such that $x = z * t$
or equivalently
$x \wedge y = x * (x \Rightarrow_* y)$.

A left continuous t-norm is continuous iff it is divisible

HÁJEK'S BASIC (FUZZY) LOGIC BL

Axioms:

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \& \psi \to \varphi$
(A3)	$\varphi \& \psi \to \psi \& \varphi$
(A4)	$\varphi \& (\varphi \to \psi) \to \psi \& (\psi \to \varphi)$
(A5a)	$(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$
(A5b)	$((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(A6)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A7)	$0 \to \varphi$

Inference Rule: Modus Ponens

Main examples: Lukasiewicz, Gödel and Product logics.

HÁJEK'S BASIC (FUZZY) LOGIC BL

Basic connectives: $\&$, $\to$, 0, 1.

Definable connectives: $\neg$, $\wedge$, $\vee$,

$$\neg\varphi \text{ means } \varphi \to 0,$$

$$\varphi \wedge \psi \text{ means } \varphi \& (\varphi \to \psi)$$

$$\varphi \vee \psi \text{ means } ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi).$$

MAIN AXIOMATIC EXTENSIONS OF BL

- Lukasiewicz BL plus

$$\neg\neg\varphi \to \varphi$$

- Gödel BL plus

$$\varphi \to \varphi \& \varphi$$

- Product BL plus

$$\phi \wedge \neg\phi = 0$$
$$\neg\neg\varphi \to ((\varphi \& \psi \to \varphi \& \chi) \to (\psi \to \chi))$$

ALGEBRAIC COUNTERPART OF BASIC (FUZZY) LOGIC BL

Algebraization: BL is algebraizable and the associated variety is the variety of BL-algebras

BL-algebra: $(L.*, \Rightarrow, \wedge, \vee, 0, 1)$ satisfying:
i) $(L.\wedge, \vee, 0, 1)$ is a distributive lattice,
ii) $(L, *, 1)$ is a commutative monoid with unit 1,
iii)$*$ and $\Rightarrow$ forms a residuated pair,i.e $x * z \leq y$ iff $x \Rightarrow y \geq z$
iv) Prelinearity $x \Rightarrow y \vee y \Rightarrow x = 1$,
v) Divisibility: $x \wedge y = x * (x \Rightarrow y)$.

BL-algebra is a prelinear and divisible residuated lattice

Example: $[0, 1]$ with usual order, a continuous t-norm and its residuum

COMPLETENESS OF HÀJEK'S BASIC (FUZZY) LOGIC BL

General completeness:
Provable formulas $= \bigcap_{L \in BL} \{\varphi \mid \varphi$ *is a tautology on L*$\}$.

Standard Completeness:
Provable formulas $= \bigcap_{*} \{\varphi \mid \varphi$ *is a tautology on* $[0,1]_*\}$
where $[0,1]_*$ means the algebra $([0,1], *, \Rightarrow_*, \neg_*, \max, \min, 0, 1)$.

Strong Standard Completeness: Let Γ a finite set of formulas,
$\Gamma \vdash_{BL} \varphi$ *iff* $\Gamma \models_{[0,1]_*} \varphi$ *for all continuous t-norm* $*$.

Lukasiewicz Logic *MV-algebras*

BL + $(\neg\neg\phi \to \phi)$ Standard complete, Rose, Roser(1958)

Gödel Logic *Gödel-algebras*

BL + $(\phi \to (\phi \& \phi))$ Standard complete, Dummett (1959)

Product Logic *Product-algebras*

BL + Standard complete, Hàjek et alt. (1996)

$\phi \wedge \neg\phi = 0$

$\neg\neg\varphi \to ((\varphi \& \psi \to \varphi \& \chi) \to (\psi \to \chi)$.

SOME INTERESTING RESULTS

Any BL-chain is an ordinal sum of MV, Godel and Product chains.

Any finite BL-chain (t-norm like divisible or 1-smooth over a finite chain) is an ordinal sum of L_n and G_k. (Mayor et alt.)

The valid equalities (involving operations of BL algebras) on L-fuzzy sets for all BL algebra L coincide with the valid equalities for all continuous t-norm and its residuum.

The variety generated by a continuous t-norm and its residuum, $\mathbf{V}([0,1]_*)$, and its corresponding logic are finite axiomatizable (Esteva, Godo, Montagna).

T-NORMS AND RESIDUATION

A t-norm $*$ has residuum $\Rightarrow_*$ iff it is left-continuous.

Order-definability: $x \leq y$ if and only if $x \Rightarrow_* y = 1$

Transitivity: $(x \Rightarrow_* y) * (y \Rightarrow_* z) \leq x \Rightarrow_* z$

Prelinearity: $(x \Rightarrow_* y) \vee (y \Rightarrow_* x) = 1$

max-def.: $\max(x, y) = \min(((x \Rightarrow_* y) \Rightarrow_* y), ((y \Rightarrow_* x) \Rightarrow_* x)))$

No-divisibility:
It satisfy only the following inequality:
$x \wedge y \geq x * (x \Rightarrow_* y)$.

MONOIDAL T-NORM BASED LOGIC (Esteva and Godo)

Axioms:

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi * \psi \to \varphi$
(A3)	$\varphi * \psi \to \psi * \varphi$
(A4)	$\varphi \wedge \psi \to \varphi$
(A5)	$\varphi \wedge \psi \to \psi \wedge \varphi$
(A6)	$\varphi * (\varphi \to \psi) \to \varphi \wedge \psi$
(A7a)	$(\varphi \to (\psi \to \chi)) \to (\varphi * \psi \to \chi)$
(A7b)	$(\varphi * \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(A8)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A9)	$0 \to \varphi$

The rule of inference of MTL is *modus ponens*.

MONOIDAL T-NORM BASED LOGIC

Basic connectives: $\&$, $\to$, $\wedge$, 0, 1.

Definable connectives: $\neg$, $\vee$,

$$\neg\varphi \text{ means } \varphi \to 0$$

$$\varphi \vee \psi \text{ means } ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$$

Theorems and proofs of the logic are obtained as usual.

ALGEBRAIC COUNTERPART OF MTL LOGIC

Algebraization: MTL is algebraizable and the associated variety is the variety of MTL-algebras

MTL-algebra: algebras $(L.*, \Rightarrow, \neg, \wedge, \vee, 0, 1)$ satisfying:
i) $(L.\wedge, \vee, 0, 1)$ is a distributive lattice,
ii) $(L, *, 1)$ is a commutative monoid with unit 1,
iii)$*$ and $\Rightarrow$ forms a residuated pair,i.e $x * z \leq y$ iff $x \Rightarrow y \geq z$
iv) Prelinearity $x \Rightarrow y \vee y \Rightarrow x = 1$,

A MTL-algebra is a residuated lattice satisfying prelinearity.

COMPLETENESS OF MTL LOGIC (Jenei-Montagna)

General Completeness:
Provable formulas $= \bigcap_{L \in MTL} \{\varphi \mid \varphi$ *is a tautology on L*$\}$.

Standard Completeness:
Provable formulas $= \bigcap_{*} \{\varphi \mid \varphi$ *is a tautology on* $[0,1]_{*}\}$
where $[0,1]_{*}$ means the algebra $([0,1], *, \Rightarrow_{*}, max, min, 0, 1)$ for any left- continuous t-norm $*$.

Strong Standard Completeness:
$\Gamma \vdash \varphi$ *iff* $\Gamma \models_{[0,1]_{*}} \varphi$ *for all left-continuous t-norm* $*$.

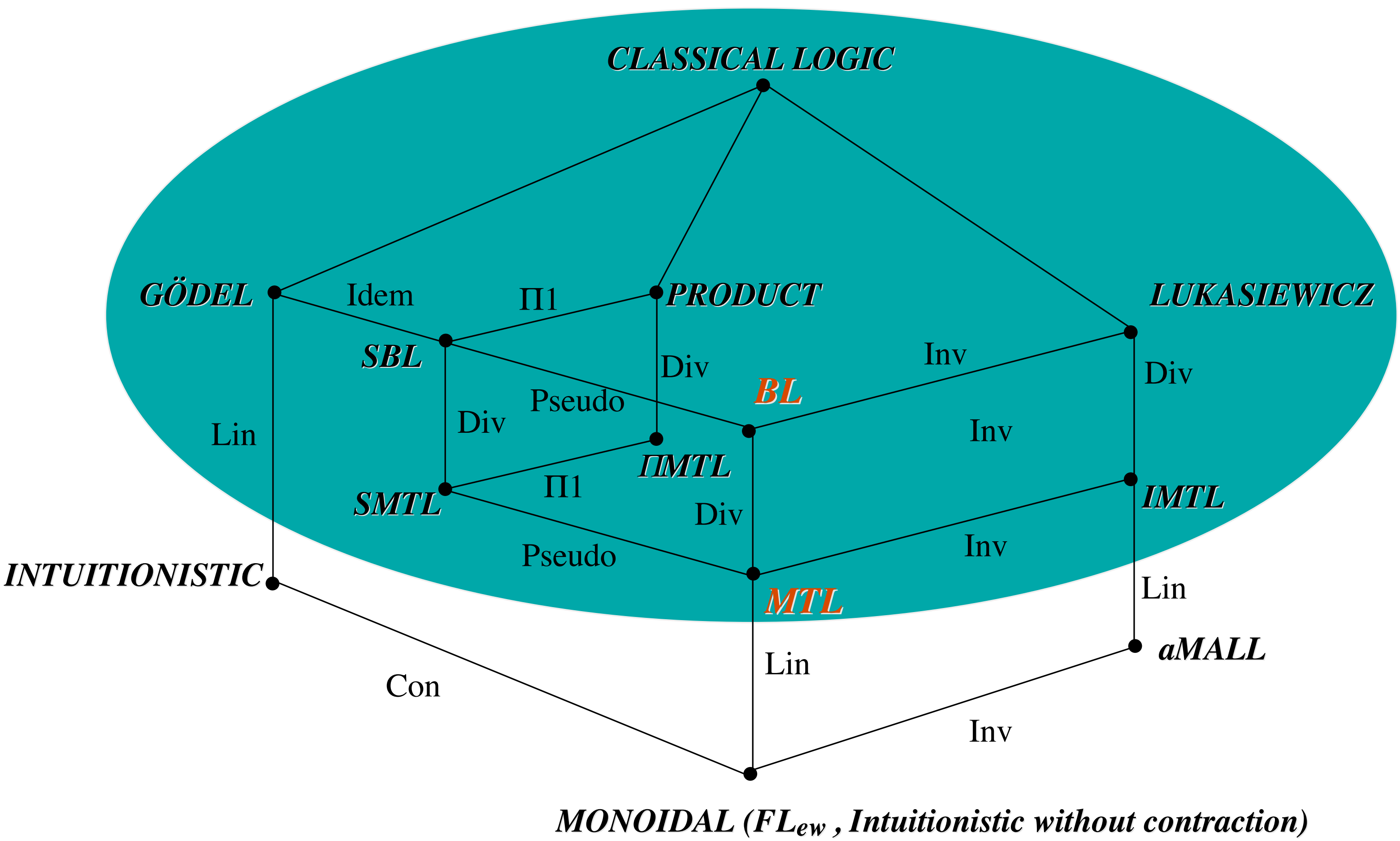
SOME INTERESTING REMARKS

There are no decomposition theorem for MTL-chains as ordinal sum (like for left-continuous t-norms) .

Any left-continuous t-norm like (having residuum) over a countable chain are embeddable in (is a restriction of) a left-continuous t-norm (over [0,1]) (Jenei- Montagna).

This result is not true for restricted subfamilies of t-norms (continuous, Lukasiewicz, Product,...)

Framework of t-norm based fuzzy logics



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SOME EXTENDED TOPICS OF THE AGENDA OF FUZZY LOGIC

1 Fuzzy Logic with "fuzzy equality". Fuzzy "If-Then rules"

2 Classical and Fuzzy quantifiers

3 Truth degrees and many-valued logic.

- F L preserving truth degrees (not truth preserving)
- F L and partial truth: adding truth degrees in the language

Fuzzy Logics preserving degrees of truth I

A formula φ is deduced from a finite set of formulas Γ if $e(\varphi) = 1$ for all evaluation e such that $e(\gamma) = 1$ for all $\gamma \in \Gamma$. Deduction is truth preserving (denoted $\Gamma \models^1 \varphi$)

Logic preserving degrees of truth

$\Gamma \models^{\leq} \varphi$ iff
for all evaluation e if $e(\gamma) \geq r$ for all $\gamma \in \Gamma$, then $e(\varphi) \geq r$, iff
for all evaluation e, $\min\{e(\gamma) \mid \gamma \in \Gamma \leq e(\varphi)$

Fuzzy Logics preserving degrees of truth II

$\Gamma \models^{\leq} \varphi$ iff $\models^{1} (\wedge\Gamma) \to \varphi$

Tautologies of L^1 coincidee with tautologies of $L^{\leq}$

From axiomatization of L^1 is possible to find an axiomatization of $L^{\leq}$.

Fuzzy Logics preserving degrees of truth II

Bou, Esteva, Font, Godo, Gispert, Torrens, Verdú,
T-norm based fuzzy logics preserving degrees of truth,
Journal of Logic and Computation 19 (2009) 10311069

SOME EXTENDED TOPICS OF THE AGENDA OF FUZZY LOGIC

1 Fuzzy Logic with "fuzzy equality". Fuzzy "If-Then rules"

2 Classical and Fuzzy quantifiers

3 Truth degrees and many-valued logic.

- F L preserving truth degrees (not truth preserving)
- F L and partial truth: adding truth degrees in the language

SOME MOTIVATING IDEAS

Fuzzy logics are logics of vague, gradual properties.

Fuzzy logics are logics of comparative degree of truth:

$$T \vdash \varphi \to \psi \text{ iff } truth(\varphi) \leq truth(\psi)$$

Generalized Modus Ponens is defined by

$\boxed{v(p \to q) \geq \alpha, \quad v(p) \geq \beta}$ imply $v(q) \geq \alpha * \beta$

INTRODUCING TRUTH-VALUES IN THE LANGUAGE

But to reason explicitly with partial degrees of truth, it is natural to *syntactically* introduce truth values in the language [Pavelka, 79],[Novák et al, 99],[Gerla, 01].

$$T \vdash \overline{r} \to \psi \text{ iff } r \leq truth(\psi)$$

Interest for knowledge representation purposes, e.g. in fuzzy description logics, fuzzy logic programming systems (weighted Horn-rules $(H \leftarrow B, r)$), fuzzy IF-THEN rules, . . .

PAVELKA LOGIC

Pavelka introduced in [Pavelka, 79] a propositional many-valued logical system, denoted here redPL, over redLukasiewicz logic by adding into the language as many truth constants as truth values (a truth constant $\overline{r}$ for each real $r \in [0,1]$) and some additional axioms.

This logic is not (strongly) complete w.r.t. the standard Lukasiewicz algebra $[0,1]_L$ (interpreting each $\overline{r}$ as the real r).

$$\{\overline{r} \to \varphi \mid r < \alpha\} \nvdash \overline{\alpha} \to \varphi$$

But, Pavelka proved that PL is indeed redcomplete in a weaker sense.

PAVELKA STYLE COMPLETENESS

Truth degree of φ in T:

$$\|\varphi\|_T = \inf\{e(\varphi) \mid e \text{ evaluation model of } T\}$$

Provability degree of φ in T:

$$|\varphi|_T = \sup\{r \mid T \vdash_{PL} \overline{r} \to \varphi\}.$$

Pavelka-style completeness: $\|\varphi\|_T = |\varphi|_T$

Moreover, if the language is extended with any further connective Pavelka-style completeness is preserved iff the corresponding truth-function on the real unit interval is a continuous (real) function.

OTHER RATIONAL EXPANSIONS

Similar *rational* expansions for other popular residuated fuzzy logics can be defined, but Pavelka-style completeness strongly relies on the continuity of Lukasiewicz truth-functions in $[0, 1]$.

Lukasiewicz logic is the only t-norm based fuzzy logic with continuous truth-functions. We cannot expect fully analogous results with expansions of other fuzzy logics with rational truth-constants.

One way to overcome the problem is to add some (infinitary) rules: For example for Rational Product logic,

$$\frac{\{\varphi \to \overline{r} \mid r \text{ rational} \in (0, 1]\}}{\varphi \to \overline{0}}$$

RATIONAL PAVELKA LOGIC (HÁJEK)

PL was significantly simplified in [Hájek, 98]. The language is expanded by countably-many truth-constants, one $\overline{r}$ for each rational $r \in [0,1]$. The logic is an extension of Lukasiewicz logic by the so-called book-keeping axioms:

$$\overline{r} \& \overline{s} \leftrightarrow \overline{r * s}$$
$$\overline{r} \to \overline{s} \leftrightarrow \overline{r \Rightarrow s}$$

where $*$ and $\Rightarrow$ denote the Lukasiewicz truth-functions.

This system, called Rational Pavelka Logic (RPL), satisfies the same Pavelka-style completeness and it is strongly complete for finite theories. Also corresponding predicate calculus RPL$\forall$ is defined.

PAVELKA STYLE EXPANSIONS: NEW APPROACH

Let $\mathcal{L}$ be an axiomatic extension of BL (MTL) and let $*$ be a (left) continuous t-norm such that $\mathcal{L}$ is complete wrt $[0,1]_*$, and $*$ (and $\Rightarrow_*$) is closed over $[0,1] \cap \mathbb{Q}$.

Propositional logic $R\mathcal{L}(*)$:

language: $\mathcal{L}^{\mathcal{R}} = \mathcal{L} \cup \{\overline{r} : r \in \mathbb{Q} \cap (0,1]\}$

axioms: those of $\mathcal{L}$ plus 'book-keeping axioms':

$$\overline{r} \& \overline{s} \leftrightarrow \overline{r * s}$$
$$\overline{r} \wedge \overline{s} \leftrightarrow \overline{\min(r,s)}$$
$$(\overline{r} \to \overline{s}) \leftrightarrow \overline{r \Rightarrow_* s}$$

PAVELKA STYLE EXPANSIONS: NEW APPROACH

Define R$\mathcal{L}(*)$-algebras as usual,
$\mathcal{A} = \langle A, \&, \to, \wedge, \vee, \{\overline{r}^{\mathcal{A}} : r \in \mathbb{Q} \cap [0,1]\}\rangle$
where $\langle A, \&, \to, \wedge, \vee, 0, 1\}\rangle$ is a $\mathcal{L}$-algebra, and the interpretation of the truth constants in $\mathcal{A}$ satisfy the book-keeping axioms.

evaluations: $e : \mathcal{L}^{\mathcal{R}} \to \mathcal{A}$ such that $e(\overline{r}) = \overline{r}^{\mathcal{A}}$

BASIC REFERENCE

Francesc Esteva; Joan Gispert; Llus Godo; Carles Noguera "Adding truth-constants to logics of continuous t-norms: axiomatization and completeness results", Fuzzy Sets and Systems, vol. 158, no. 6, pp. 597-618

GENERAL RESULTS (Esteva, Godo, Noguera)

- $R\mathcal{L}(*)$ is an algebraizable logic whose equivalent algebraic semantics is the variety of $R\mathcal{L}(*)$-algebras.

- $R\mathcal{L}(*)$ is complete with respect the class of $R\mathcal{L}(*)$-chains.

- Conservativeness of $R\mathcal{L}(*)$ with respect to $\mathcal{L}$.

Question: what about standard completeness . . .

RESULTS FOR EXPANSIONS OF LUKASIEWICZ, GÖDEL AND PRODUCT LOGICS WITH TRUTH-CONSTANTS

(Savický, Cignoli, Esteva, Godo, Noguera)

- The expansion of Lukasiewicz with truth-values is strong standard complete for finite theories (Hájek's book)
  $\Gamma \vdash_{RL} \varphi$ iff $\Gamma \models_{[0,1]_{RL}} \varphi$

- The expansions of Gödel and Product logics with truth-values are standard complete (Theorems of the logic coincides with tautologies over the standard algebra, $\vdash_{RG} \varphi$ iff $\models_{[0,1]_{RG}} \varphi$) but not strong standard complete even for finite theories.

RESULTS FOR EVALUATED FORMULAS

• Strong standard completeness for finite theories is only true for Lukasiewicz logic. For Gödel and Product it is recovered for the restricted language of evaluated formulas
(formulas of type $\overline{r} \to \varphi$ where φ is a formula without constants)
$\Gamma \vdash_{RG} \overline{r} \to \varphi$ IFF $\Gamma \models_{[0,1]_{RG}} \overline{r} \to \varphi$

• Remember that evaluated formulas are interesting for knowledge representation purposes, e.g. in fuzzy description logics, fuzzy logic programming systems (weighted Horn-rules ($H \leftarrow B, r$)), fuzzy IF-THEN rules, . . .
and in all these cases knowledge is represented by a finite number of evaluated formulas

SUMMARY

Fuzzy (residuated) multiple-valued logics are the core of Fuzzy Logic in narrow sense. Researchers in logic are studying different fuzzy logical systems.

An interesting job is to apply results and to interpret the concepts of the fuzzy agenda in this logical setting. I have showed some examples but the job is just initiated.

FUZZY LOGIC IS BEAUTIFUL AND INTERESTING

Thank you for your attention