# Jan Laštovička and Lhouari Nourine: Completations with distributive concept lattice

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## Introduction

Formal concept analysis is a method dealing with binary object-attribute data. It extracts clusters called formal concepts and orders them by specialization-generalization relation. With this ordering, concepts form a lattice structure called concept lattice.

Practice, data are often incomplete. They can be completed in more ways and each one gives different concept lattice. One can have an information that the concept lattice of a completion of the incomplete data is distributive. This information restricts the possibilities of completions of the data.

This poster deals with question how to determine completions of incomplete data such that the concept lattice is distributive. This generalizes the work of Felsner *et. al.*[1] where the distributive lattice is a chain. They also show that the set of all chain-completions forms a lattice, i.e. chain-completions are in bijection with maximal chains of a lattice.

### Formal concept analysis

formal context: $\langle X,Y,I\rangle$, $I \subseteq X\times Y$ pair of induced mappings:

$$A^{\uparrow} = \{y \in Y \mid \langle x,y\rangle \in I \text{ for all } x \in A\}$$
$$B^{\downarrow} = \{x \in X \mid \langle x,y\rangle \in I \text{ for all } y \in B\}$$

concept lattice:
:: $\mathscr{B}(X,Y,I) = \{\langle A,B\rangle \in 2^X \times 2^Y \mid A^{\uparrow} = B \text{ and } B^{\downarrow} = A\}$

:: partial order:
$\langle A_1,B_1\rangle \leq \langle A_2,B_2\rangle$ iff $A_1 \subseteq A_2$ (iff $B_2 \subseteq B_1$)

### Arrow relations

$\langle X,Y,I\rangle \ldots$ formal context
$x \in X$ and $y \in Y$

$$x \swarrow_I y \quad \text{if } \begin{cases} \langle x,y\rangle \notin I \text{ and} \\ \{x\}^{\uparrow} \subset \{x_1\}^{\uparrow} \text{ implies } \langle x_1,y\rangle \in I \end{cases}$$
$$x \nearrow_I y \quad \text{if } \begin{cases} \langle x,y\rangle \notin I \text{ and} \\ \{y\}^{\downarrow} \subset \{y_1\}^{\downarrow} \text{ implies } \langle x,y_1\rangle \in I \end{cases}$$
$$x \swarrow\!\!\!\nearrow_I y \quad \text{if } x \swarrow_I y \text{ and } x \nearrow_I y$$

distributive laws:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

For a finite concept lattice $\mathscr{B}(X,Y,I)$, the following conditions are equivalent:
:: $\mathscr{B}(X,Y,I)$ is distributive.

:: From $x \swarrow\!\!\!\nearrow y_1$ and $x \nearrow y_2$ it follows that $\{y_1\}^{\downarrow} = \{y_2\}^{\downarrow}$.

:: From $x_1 \swarrow\!\!\!\nearrow y$ and $x_2 \swarrow y$ it follows that $\{x_1\}^{\uparrow} = \{x_2\}^{\uparrow}$.

## Incomplete data

### Incomplete contexts

Boolean algebra with variables $u_1,\ldots,u_k$:
:: $\mathbf{L}\ldots$finite Boolean algebra

:: $U = \{u_1,\ldots,u_k\}\ldots$set of variables

:: $\iota: U \to L$, such that $\mathbf{L}$ is generated by $\iota(U)$
We will identify variables $U$ with their image under mapping $\iota$.
Elements of $\mathbf{L}$ are interpreted as conditions built from variables by operations of Boolean algebra.

incomplete $\mathbf{L}$-context: $\langle X,Y,I\rangle$ where $I \in L^{X\times Y}$ such that $I(X\times Y) \subseteq U \cup \{0,1\}$
$v: U \to \mathbf{2}\ldots$ assignment
If $v$ can be extended to a homomorphism $\bar{v}: \mathbf{L} \to \mathbf{2}$, it is called admissible.

$v\ldots$ admissible assignment
Note $\langle X,Y,\bar{v}\circ I\rangle$ is a classical formal context (completion) and $\mathscr{B}(X,Y,\bar{v}\circ I)$ is its concept lattice.

### Arrow L-relations

$\mathbf{L}\ldots$ Boolean algebra with variables
$A,B \in L^X$
Strict containment: $S_{\subset}(A,B) = S(A,B) \wedge \big(S(B,A)\big)'$

$\langle X,Y,I\rangle\ldots$ incomplete $\mathbf{L}$-context
We define binary $\mathbf{L}$-relations $\nearrow, \swarrow, \swarrow\!\!\!\nearrow \in L^{X\times Y}$ by

$$x \swarrow_I y = I(x,y)' \wedge \bigwedge_{x_1\in X} S_{\subset}(\{x\}^{\uparrow},\{x_1\}^{\uparrow}) \to I(x_1,y)$$
$$x \nearrow_I y = I(x,y)' \wedge \bigwedge_{y_1\in Y} S_{\subset}(\{y\}^{\downarrow},\{y_1\}^{\downarrow}) \to I(x,y_1)$$
$$x \swarrow\!\!\!\nearrow_I y = (x \swarrow_I y) \wedge (x \nearrow_I y)$$

**Theorem 1** *For an incomplete $\mathbf{L}$-context with finite $X$ and $Y$, and an admissible assignment $v$ the following conditions are equivalent:*
:: *$\mathscr{B}(X,Y,\bar{v}\circ I)$ is distributive.*
::
$$\bar{v}\Big(\bigwedge_{\substack{x\in X\\ y_1,y_2\in Y}} (x \swarrow\!\!\!\nearrow_I y_1) \wedge (x \nearrow_I y_2) \to (\{y_1\}^{\downarrow_I} \approx \{y_2\}^{\downarrow_I})\Big) = 1$$
::
$$\bar{v}\Big(\bigwedge_{\substack{x_1,x_2\in X\\ y\in Y}} (x_1 \swarrow\!\!\!\nearrow_I y) \wedge (x_2 \swarrow_I y) \to (\{x_1\}^{\uparrow_I} \approx \{x_2\}^{\uparrow_I})\Big) = 1$$

## Example

Consider incomplete data:

| | $y_1$ | $y_2$ | $y_3$ |
|---|---|---|---|
| $x_1$ | $u_1$ | | |
| $x_2$ | | $u_2$ | $u_3$ |
| $x_3$ | | | $\times$ |

Variables: $U = \{u_1,u_2,u_3\}$

Consider assignments:
$v_1 = \emptyset, v_2 = \{u_1,u_3\}, v_3 = \{u_1,u_2\}, v_4 = \{u_1,u_2,u_3\}$

Define Boolean algebra with variables $u_1,u_2,u_3$:
:: $\mathbf{L} = \mathbf{2}^{\{v_1,v_2,v_3,v_4\}}$

:: $(\iota(u))(v) = v(u)$
Admissible assignments are precisely $v_1,v_2,v_3,v_4$.

Data can be formalized as incomplete $\mathbf{L}$-context $\langle X,Y,I\rangle$.
We have four possible completions (with arrow relations):

$\langle X,Y,\bar{v}_1\circ I\rangle$

| | $y_1$ | $y_2$ | $y_3$ |
|---|---|---|---|
| $x_1$ | | | $\swarrow\!\!\!\nearrow$ |
| $x_2$ | | | $\swarrow\!\!\!\nearrow$ |
| $x_3$ | $\swarrow\!\!\!\nearrow$ | $\swarrow\!\!\!\nearrow$ | $\times$ |

$\langle X,Y,\bar{v}_2\circ I\rangle$

| | $y_1$ | $y_2$ | $y_3$ |
|---|---|---|---|
| $x_1$ | $\times$ | $\swarrow$ | $\swarrow\!\!\!\nearrow$ |
| $x_2$ | $\swarrow\!\!\!\nearrow$ | $\swarrow$ | $\times$ |
| $x_3$ | $\swarrow\!\!\!\nearrow$ | $\swarrow$ | $\times$ |

$\langle X,Y,\bar{v}_3\circ I\rangle$

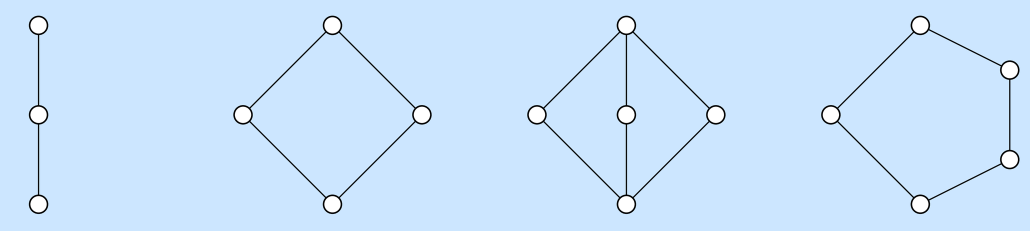
| | $y_1$ | $y_2$ | $y_3$ |
|---|---|---|---|
| $x_1$ | $\times$ | $\swarrow\!\!\!\nearrow$ | $\swarrow\!\!\!\nearrow$ |
| $x_2$ | $\swarrow\!\!\!\nearrow$ | $\times$ | $\swarrow\!\!\!\nearrow$ |
| $x_3$ | $\swarrow\!\!\!\nearrow$ | $\swarrow\!\!\!\nearrow$ | $\times$ |

$\langle X,Y,\bar{v}_4\circ I\rangle$

| | $y_1$ | $y_2$ | $y_3$ |
|---|---|---|---|
| $x_1$ | $\times$ | $\swarrow$ | $\swarrow\!\!\!\nearrow$ |
| $x_2$ | $\swarrow\!\!\!\nearrow$ | $\times$ | $\times$ |
| $x_3$ | $\nearrow$ | $\swarrow\!\!\!\nearrow$ | $\times$ |

Their concept lattices:

$\mathscr{B}(X,Y,\bar{v}_1\circ I)$ $\mathscr{B}(X,Y,\bar{v}_2\circ I)$ $\mathscr{B}(X,Y,\bar{v}_3\circ I)$ $\mathscr{B}(X,Y,\bar{v}_4\circ I)$



It can be checked from completion whether it has distributive concept lattice. Concept lattices $\mathscr{B}(X,Y,\bar{v}_1\circ I)$ and $\mathscr{B}(X,Y,\bar{v}_2\circ I)$ are distributive but $\mathscr{B}(X,Y,\bar{v}_3\circ I)$ and $\mathscr{B}(X,Y,\bar{v}_4\circ I)$ are not.

We can obtain this information directly from the incomplete context using Theorem 1. First we will compute arrow $\mathbf{L}$-relations:

| $\nearrow$ | $y_1$ | $y_2$ | $y_3$ |
|---|---|---|---|
| $x_1$ | $0$ | $u_2 \wedge u_3'$ | $1$ |
| $x_2$ | $u_1$ | $0$ | $u_3'$ |
| $x_3$ | $1$ | $u_2 \vee u_1'$ | $0$ |

| $\swarrow$ | $y_1$ | $y_2$ | $y_3$ |
|---|---|---|---|
| $x_1$ | $0$ | $u_1$ | $1$ |
| $x_2$ | $u_2 \vee u_3$ | $u_2' \wedge u_3$ | $u_3'$ |
| $x_3$ | $u_2' \vee u_3'$ | $1$ | $0$ |

| $\swarrow\!\!\!\nearrow$ | $y_1$ | $y_2$ | $y_3$ |
|---|---|---|---|
| $x_1$ | $0$ | $u_2 \wedge u_3'$ | $1$ |
| $x_2$ | $u_1$ | $0$ | $u_3'$ |
| $x_3$ | $u_2' \vee u_3'$ | $u_2 \vee u_1'$ | $0$ |

Second we will compute values
$(x \swarrow\!\!\!\nearrow_I y_1) \wedge (x \nearrow_I y_2) \to (\{y_1\}^{\downarrow_I} \approx \{y_2\}^{\downarrow_I})$:

| | $y_1,y_2$ | $y_2,y_1$ | $y_1,y_3$ | $y_3,y_1$ | $y_2,y_3$ | $y_3,y_2$ |
|---|---|---|---|---|---|---|
| $x_1$ | $1$ | $1$ | $1$ | $1$ | $u_2' \vee u_3$ | $u_2' \vee u_3$ |
| $x_2$ | $1$ | $1$ | $u_2' \vee u_3$ | $u_2' \vee u_3$ | $1$ | $1$ |
| $x_3$ | $u_2' \vee u_3$ | $u_2'$ | $1$ | $1$ | $1$ | $1$ |

Observe that $\{y_1\}^{\downarrow} \approx \{y_2\}^{\downarrow} = u_1'$.
Note values for $y_1 = y_2$ are equal to 1 since $\{y\}^{\downarrow_I} \approx \{y\}^{\downarrow_I} = 1$.

Finally we compute value

$$\bigwedge_{\substack{x\in X\\ y_1,y_2\in Y}} (x \swarrow\!\!\!\nearrow_I y_1) \wedge (x \nearrow_I y_2) \to (\{y_1\}^{\downarrow_I} \approx \{y_2\}^{\downarrow_I}) = u_2'.$$

Value $u_2'$ is interpreted as a condition for an admissible assignment to produce the completion with the distributive concept lattice. Admissible assignments $v_1$ and $v_2$ satisfy this condition ($\bar{v}_1(u_2') = \bar{v}_2(u_2') = 1$) but $v_3$ and $v_4$ not ($\bar{v}_3(u_2') = \bar{v}_4(u_2') = 0$).

As a result we obtain that all completions with distributive concept lattice are $\langle X,Y,\bar{v}_1\circ I\rangle$ and $\langle X,Y,\bar{v}_2\circ I\rangle$.

## Open questions

Many opens questions remain open:
:: characterize the poset of all distributive completions, i.e. can we generalize the result of [1],

:: is it NP-complete to find a distributive completion, what about the case for distributive completions that are product of chains.

## References

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