E. Bartl, J. Konecny: Fuzzy Concepts with Rough Intents:

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Introduction

We consider a formal fuzzy concept as a collection of objects accompanied with two collections of attributes—those which are shared by all the objects and those which at least one object has. We define concept-forming operators for such objects and show their properties and their relationship to both antitone and isotone concept-forming operators.

Formal Fuzzy Conceptual Analysis

A complete residuated lattice is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that

:: $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice,

:: $\langle L, \otimes, 1\rangle$ is a commutative monoid,

:: $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$.

An $\mathbf{L}$-set (or fuzzy set) A in a universe set X is a mapping $A : X \rightarrow L$.
The set of all $\mathbf{L}$-sets in a universe X is denoted L^X.
The operations with $\mathbf{L}$-sets are defined componentwise: For instance, the intersection of $\mathbf{L}$-sets $A, B \in L^X$ is an $\mathbf{L}$-set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$, etc.

Binary $\mathbf{L}$-relations (binary fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$.

Consider the following pairs of operators induced by an $\mathbf{L}$-context $\langle X, Y, I\rangle$. First, the pair $\langle \uparrow, \downarrow\rangle$ of operators $^{\uparrow} : L^X \rightarrow L^Y$ and $^{\downarrow} : L^Y \rightarrow L^X$ is defined by

$$A^{\uparrow}(y) = \bigwedge_{x\in X} A(x) \rightarrow I(x, y),$$
$$B^{\downarrow}(x) = \bigwedge_{y\in Y} B(y) \rightarrow I(x, y).$$

Second, the pair $\langle \cap, \cup\rangle$ of operators $^{\cap} : L^X \rightarrow L^Y$ and $^{\cup} : L^Y \rightarrow L^X$ is defined by

$$A^{\cap}(y) = \bigvee_{x\in X} A(x) \otimes I(x, y),$$
$$B^{\cup}(x) = \bigwedge_{y\in Y} I(x, y) \rightarrow B(y),$$

Fixpoints of these operators are called formal concepts. The set of all formal concepts (along with set inclusion) forms a complete lattice, called $\mathbf{L}$-concept lattice.

We denote the sets of all concepts (as well as the corresponding $\mathbf{L}$-concept lattice) by $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ and $\mathcal{B}^{\cap\cup}(X, Y, I)$, i.e.

$$\mathcal{B}^{\uparrow\downarrow}(X, Y, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\uparrow} = B,\ B^{\downarrow} = A\},$$
$$\mathcal{B}^{\cap\cup}(X, Y, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\cap} = B,\ B^{\cup} = A\}.$$

Results

We consider concept-forming operators induced by $\mathbf{L}$-context $\langle X, Y, I\rangle$ defined as follows:

Let $\langle X, Y, I\rangle$ be an $\mathbf{L}$-context. Define rough fuzzy concept-forming operators as

$$A^{\triangle} = \langle A^{\uparrow}, A^{\cap}\rangle$$
$$\langle B_1, B_2\rangle^{\triangledown} = B_1^{\downarrow} \cap B_2^{\cup}$$

for $A \in L^X, B_1, B_2 \in L^Y$. Rough fuzzy concept is then a fixed point of $\langle \triangle, \triangledown\rangle$, i.e. a pair $\langle A, \langle B_1, B_2\rangle\rangle \in L^X \times (L \times L)^Y$ such that $A^{\triangle} = \langle B_1, B_2\rangle$ and $\langle B_1, B_2\rangle^{\triangledown} = A$. $A^{\uparrow}$ and $A^{\cap}$ are called universal and existential intent, respectively.

That means, $^{\triangle}$ gives intents w.r.t. both $\langle \uparrow, \downarrow\rangle$ and $\langle \cap, \cup\rangle$; $^{\triangledown}$ then gives intersection of extents related to the corresponding intents.

Properties of rough concept-forming operators

Proposition: Set of all fixed-points of $\langle \triangle, \triangledown\rangle$ together with $\preceq$ defined as

$$\langle A, B_1, B_2\rangle \preceq \langle A, B_1, B_2\rangle = S(A, A')$$
$$= S(B_1', B_1) \wedge S(B_2, B_2')$$

forms a completely lattice $\mathbf{L}$-ordered set.

Proposition: For natural $A \in L^X$ (i.e. A contains at least one element in degree 1), we have $A^{\uparrow} \subseteq A^{\cap}$, for crisp singleton $A \in L^X$, we have $A^{\uparrow} = A^{\cap}$.

Proposition: For $\mathcal{S} \subseteq L^X$, let $[\mathcal{S}]$ denote an $\mathbf{L}$-closure span of $\mathcal{S}$, i.e. the smallest $\mathbf{L}$-closure system containing $\mathcal{S}$. We have

$$[\mathrm{Ext}^{\uparrow\downarrow}(X, Y, I) \cup \mathrm{Ext}^{\cap\cup}(X, Y, I)] = \mathrm{Ext}^{\triangle\triangledown}(X, Y, I).$$

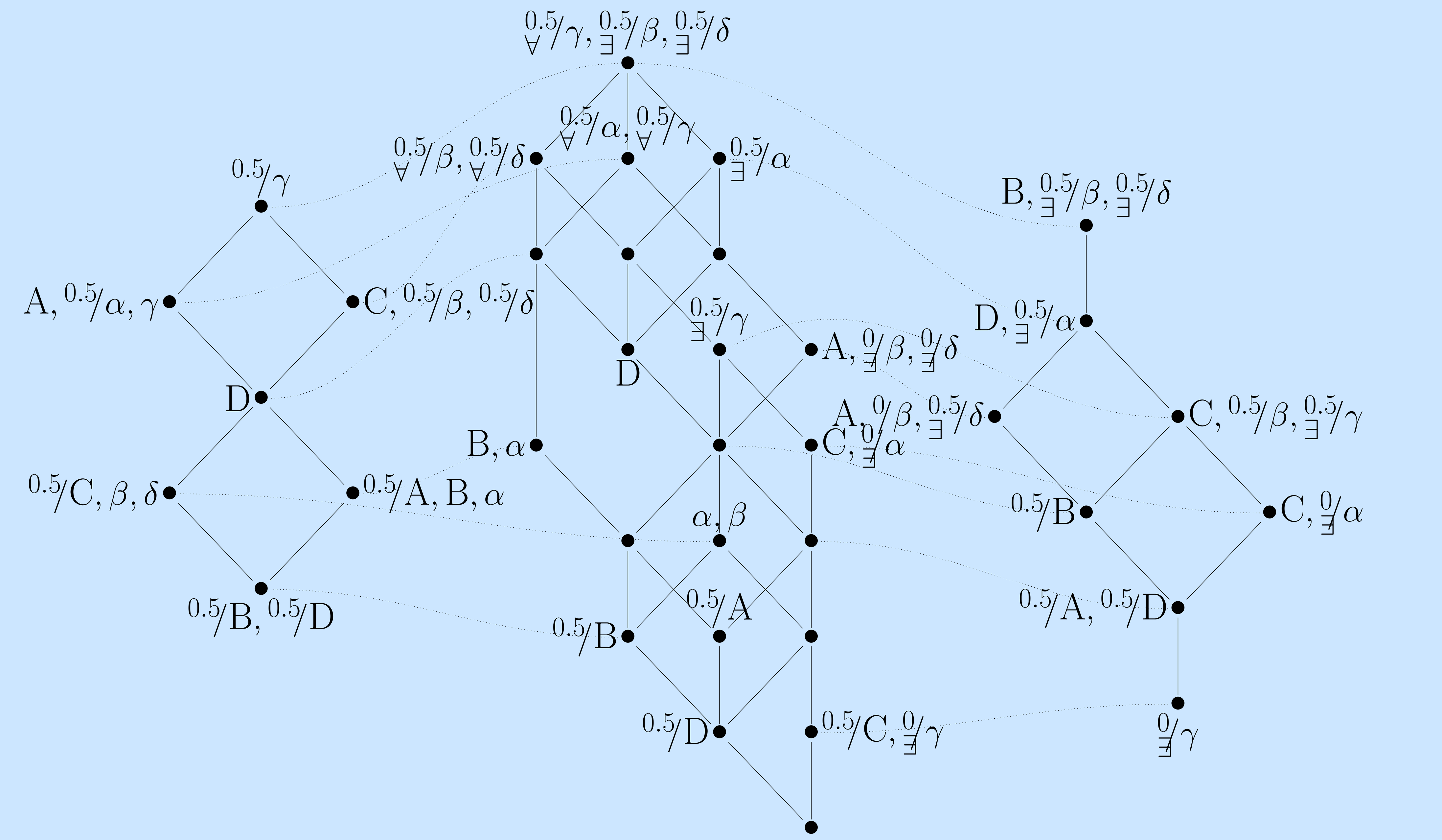
From Theorem **??** one can observe that no extent is lost in comparison with $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ and $\mathcal{B}^{\cap\cup}(X, Y, I)$.
Proposition: $\mathrm{Ext}^{\uparrow\downarrow}(X, Y, I) \subseteq \mathrm{Ext}^{\triangle\triangledown}(X, Y, I)$ and $\mathrm{Ext}^{\cap\cup}(X, Y, I) \subseteq \mathrm{Ext}^{\triangle\triangledown}(X, Y, I)$.

In addition, no concept is lost:

Proposition: For each $\langle A, B_1\rangle \in \mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ there is $\langle A, \langle B_1, A^{\cap}\rangle\rangle \in \mathcal{B}^{\triangle\triangledown}(X, Y, I)$.
For each $\langle A, B_2\rangle \in \mathcal{B}^{\cap\cup}(X, Y, I)$ there is $\langle A, \langle A^{\uparrow}, B_2\rangle\rangle \in \mathcal{B}^{\triangle\triangledown}(X, Y, I)$.

Remark: Note that new extents, i.e. extents not present in $\mathrm{Ext}^{\uparrow\downarrow}(X, Y, I) \cup \mathrm{Ext}^{\cap\cup}(X, Y, I)$, can appear in $\mathrm{Ext}^{\triangle\triangledown}(X, Y, I)$; one can observe this fact in Picture.

Picture



Rough Approximations

Let $\langle X, Y, I\rangle$ be an $\mathbf{L}$-context, E be an $\mathbf{L}$-equivalence on Y. Define rough concept-forming operators as follows:

$$A^{\triangle_E} = \langle A^{\uparrow\Downarrow_E}, A^{\cap\Uparrow_E}\rangle,$$
$$\langle B_1, B_2\rangle^{\triangledown_E} = B_1^{\downarrow} \cap B_2^{\cup}.$$

Proposition: Both, existential and universal intents are compatible with E.

The following theorem shows that a use of a rougher $\mathbf{L}$-equivalence relation leads to reduction of size of the rough $\mathbf{L}$-concept lattices. Furthermore, this reduction is natural, i.e. it preserves extents.

Proposition: Let $\langle X, Y, I\rangle$ be an $\mathbf{L}$-context, and E_1, E_2 be $\mathbf{L}$-equivalences on Y, such that $E_1 \subseteq E_2$. Then

$$\mathrm{Ext}^{\triangle_{E_1}\triangledown_{E_1}}(X, Y, I) \subseteq \mathrm{Ext}^{\triangle_{E_1}\triangledown_{E_1}}(X, Y, I).$$

Our Future Research

Our future research includes:

:: Generalization of the current setting. Note that the operators $^{\uparrow}$ and $^{\cap}$ which compute the universal and the existential intent, respectively, need not be induced by the same relation to keep main properties of the concept-forming operators.

:: This can provide interesting solution of problem of missing values in a formal fuzzy context—the idea is to use $^{\uparrow}$ induced by the context with missing values substitued by 0, and $^{\cap}$ induced by the context with missing values substitued by 1.

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