Ivan Chajda and Miroslav Kolařík: Orthomodular directoids

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Introduction

By a **logical structure of a physical system** (see [1,2,8] or [14]) is meant a couple $(L;F)$, where L is a nonvoid set and F is a set of functions from L into the interval $[0,1]$ of real numbers satisfying the following axioms:

- If $p,q \in L$ and $f(p)=f(q)$ for every $f \in F$ then $p=q$.
- There exists an element $u \in L$ such that $f(u)=1$ for each $f \in F$.
- For each $p \in L$, there exists an element $p' \in L$ such that $f(p)+f(p')=1$ for every $f \in F$.

Let $\leq$ be the relation defined on L by

$$p \leq q \quad \text{if and only if} \quad f(p) \leq f(q) \ \text{for every} \ f \in F.$$

Then $\leq$ is a partial order on L with the least and greatest element. We say that $p,q \in L$ are **orthogonal** if $p \leq q'$ (which is equivalent to $q \leq p'$, see [1] for details).

We add one more axiom:

- For every orthogonal elements $p,q \in L$ there exists supremum $s = \sup(p,q)$ and $f(s)=f(p)+f(q)$ for each $f \in F$.

It is well-known that the system $(L;\leq,',0,1)$ is an orthomodular poset, the so-called **associated poset with the logical structure** $(L;F)$, see e.g. [1]. Hence, orthomodular posets serve as an axiomatic description of physical systems, see e.g. [6,7]. If $\sup(p,q)$ exists for each couple p,q of elements of L, then $(L;\leq,',0,1)$ becomes an orthomodular lattice. Hence, the theory of orthomodular posets includes the theory of orthomodular lattices and, simultaneously, serves as an axiomatization of the logic of physical systems. In particular, it axiomatizes the logic of quantum mechanics, see [2,7,8,11] and [14].

Due to the above mentioned properties, orthomodular posets were and are studied by numerous authors for several decades see e.g. [6,7,9,13,14]. However, up to now, orthomodular posets were treated as partial algebras where the binary operation of supremum is ensured only for orthogonal or comparable elements. We try another approach, namely to introduce a certain everywhere defined algebra which can be assigned to every orthomodular poset in the way that the underlying poset coincides with the original one but its axioms can be expressed as identities. Hence, the class of these so-called orthomodular directoids forms a variety of algebras having nice algebraic properties. Moreover, every orthomodular poset can be recovered by means of this assigned algebra despite the fact that the assignment need not be done in a unique way.

Orthomodular directoids

Recall by [9] (see also [5]) that a groupoid $(A;+)$ is called a **commutative directoid** if it satisfies the following axioms:

- $x+x=x$
- $x+y=y+x$
- $x+((x+y)+z)=(x+y)+z$.

In what follows, we enrich the commutative directoid by a unary operation (orthocomplementation) and by two constants to get an algebra for our study. Since we need ask two more properties connected with orthomodular posets (namely the orthomodular law and the existence of suprema for orthogonal elements), we add two more axioms which caused that some other axioms for orthomodular directoids can follow from the remaining ones. Hence, we can define:

Definition 1

By an **orthomodular directoid** is called an algebra $\mathcal{D}=(D;+,',0,1)$ of type $(2,1,0,0)$ satisfying the following axioms:

(D1) $x+y=y+x$
(D2) $x+((x+y)+z)=(x+y)+z$
(D3) $x+0=x$
(D4) $x+x'=1$
(D5) $(((x+z)+(y+z)')'+(y+z)')+z'=z'$
(D6) $x+(x+(x+y)')'=x+y$.

Theorem 1

The axioms (D1)–(D6) are independent.

Lemma

Let $\mathcal{D}=(D;+,',0,1)$ be an orthomodular directoid. Define a binary relation $\leq$ on D as follows

$$x \leq y \quad \text{if and only if} \quad x+y=y.$$

Then $\leq$ is a partial order on D (the so called **induced order**).

Now, we recall the concept of orthomodular poset (from [1]).

Definition 2

By an **orthomodular poset** is meant a structure $\mathcal{P}=(P;\leq,',0,1)$, where $\leq$ is a partial order on P, $0 \leq x \leq 1$ for each $x \in P$, $x''=x$, x' is a complement of x and $x \leq y$ implies $y' \leq x'$, and satisfying the following two conditions:

- if $x \leq y'$ then the set $\{x,y\}$ has the supremum $x \vee y$ in $(P;\leq)$
- if $x \leq y$ then $x \vee (x \vee y')' = y$.

A representation of orthomodular posets

Example

See [1]. Let M be a finite set with an even number of elements. Let P be the set of all subsets of M which have even number of elements ordered by inclusion and let $A' = M \setminus A$, the set-theoretical complementation. Then $\mathcal{P}=(P;\subseteq,',\emptyset,M)$ is an orthomodular poset. If $|M| \geq 6$ then $\mathcal{P}$ is not a lattice.

Now we state two main theorems.

Theorem 2

Let $\mathcal{D}=(D;+,',0,1)$ be an orthomodular directoid and $\leq$ be its induced order. Then $\mathcal{P}(D)=(D;\leq,',0,1)$ is an orthomodular poset where for orthogonal elements $x,y \in D$ we have $x+y=x \vee y$.

Theorem 3

Let $\mathcal{P}=(P;\leq,',0,1)$ be an orthomodular poset. Define a binary operation $+$ on P as follows:

- $x+y=x \vee y$ if $x \vee y$ exists
- $x+y=y+x$ is an arbitrary element of $U(x,y)=\{z \in P; x,y \leq z\}$ otherwise.

Then $\mathcal{D}(P)=(P;+,',0,1)$ is an orthomodular directoid.

By Theorem 3, to every orthomodular poset $\mathcal{P}=(P;\leq,',0,1)$ can be assigned an everywhere defined algebra which is an orthomodular directoid $\mathcal{D}(P)=(P;+,',0,1)$. By Theorem 2, to the orthomodular directoid $\mathcal{D}(P)$ can be assigned an orthomodular poset $\mathcal{P}(\mathcal{D}(P))$. Since the underlying posets $(P;\leq)$ coincide in all $\mathcal{P}$, $\mathcal{D}(P)$ and $\mathcal{P}(\mathcal{D}(P))$ and the complementation is also the same, we conclude that $\mathcal{P}=\mathcal{P}(\mathcal{D}(P))$. Hence, although the directoid $\mathcal{D}(P)$ need not be assigned in a unique way, it bears all the information on $\mathcal{P}$ because $\mathcal{P}=\mathcal{P}(\mathcal{D}(P))$ for every such a directoid.
On the contrary, if $\mathcal{D}=(D;+,',0,1)$ is an orthomodular directoid, $\mathcal{P}(D)$ the assigned orthomodular poset and $\mathcal{D}(\mathcal{P}(D))$ the assigned orthomodular directoid then $\mathcal{D}$ and $\mathcal{D}(\mathcal{P}(D))$ need not be even isomorphic because the operation $+$ in $\mathcal{D}(\mathcal{P}(D))$ can be choosen differently than that in $\mathcal{D}$.

Theorem 4

Let $\mathcal{D}=(D;+,',0,1)$ be an orthomodular directoid, $\leq$ its induced order and $a \in D$. Then $([a,1];+,^a,a,1)$ for $x^a = x'+a$ is an orthomodular directoid.

The variety of orthomodular directoids

By Theorems 2 and 3, orthomodular posets can be represented by everywhere defined algebras, i.e. by orthomodular directoids. However, by Definition 1, these directoids are determined by the identities (D1)–(D6) and hence the class $\mathcal{K}$ of orthomodular directoids forms a variety of algebras. In what follows, we present several important properties of the variety $\mathcal{K}$.

Theorem 5

The variety $\mathcal{K}$ of orthomodular directoids is congruence regular and arithmetical.

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