# Petr Osicka: Triadic attribute implications: entailment and completeness

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## Introduction

We study conditional attribute implications, a particular form of attribute dependencies in three-way relational data describing objects by binary attributes they have under conditions. We focus on axiomatization of their semantical entailment and provide the completeness theorem.

### Preliminaries

A triadic context

- is a quadruple $\langle X, Y, Z, I\rangle$ where
- $X$, $Y$, and $Z$ are non-empty sets, and
- $I$ is a ternary relation between $X$, $Y$, and $Z$, i.e. $I \subseteq X \times Y \times Z$.
- The sets $X$, $Y$, and $Z$ are interpreted as the sets of objects, attributes, and conditions, respectively;
- $I$ is interpreted as the incidence relation ("to have-under relation"). That is, $\langle x, y, z\rangle \in I$ is interpreted as: "object $x$ has attribute $y$ under condition $z$".

Concept forming operators

- mappings assigning to pairs of sets (in different dimensions) a set (in the remaining dimension)
- For $C_i \subseteq X_i$ and $C_j \subseteq X_j$ we have

$$C_i^{(i,j,C_k)} = \{x_j \in X_j : \langle x_i, x_j\rangle \in I^{ij}_{C_k} \text{ for each } x_i \in C_i\},$$
$$C_j^{(i,j,C_k)} = \{x_i \in X_i : \langle x_i, x_j\rangle \in I^{ij}_{C_k} \text{ for each } x_j \in C_j\}.$$

A triadic concept

- a triplet $\langle C_1, C_2, C_3\rangle$ of $C_1 \subseteq X_1$, $C_2 \subseteq X_2$, and $C_3 \subseteq X_3$,
- for every $\{i, j, k\} = \{1, 2, 3\}$ we have

$$C_i = C_j^{(i,j,C_k)},\ C_j = C_k^{(j,k,C_i)}, \text{ and } C_k = C_i^{(k,i,C_j)}.$$

The set of all triadic concepts of $\langle X_1, X_2, X_3, I\rangle$is called the concept trilattice of $\langle X_1, X_2, X_3, I\rangle$. Note that complete trilattices are the appropriate generalizations of complete lattices (i.e. dyadic lattices) that result as naturally structured sets of fixpoints of the connections induced by ternary relations.

## Attribute implications

We start by a definition of a conditional attribute implication (Ganter & Obiedkov).

**Definition 1.** Given a triadic context $K = (X_1, X_2, X_3)$, a conditional attribute implication is a formula

$$A \stackrel{C}{\Rightarrow} B$$

where $A, B \subseteq X_2$ are sets of attributes and $C \subseteq X_3$ is a set of conditions.

The intended meaning of $A \stackrel{C}{\Rightarrow} B$ being true in a triadic context is the following:

*For every condition $c \in C$ holds: if an object has all attributes of $A$, then he has all attributes of $B$.*

This intention leads us to consider a model of conditional attribute implication as a pair of sets consisting of a set of attributes, and a set of conditions.

**Definition 2.** A triadic attribute implication $A \stackrel{C}{\Rightarrow} B$ is valid in a model $(M_1, M_2)$, $M_1 \subseteq X_2$ is a set of attributes, $M_2 \subseteq X_3$ is a set of conditions, iff

$$M_2 \subseteq C \text{ and } A \subseteq M_1 \text{ implies } B \subseteq M_2.$$

We denote it as $||A \stackrel{C}{\Rightarrow} B||_{(M_1,M_2)} = 1$

**Definition 3.** Set $T$ of conditional attribute implications is called a theory. Then Mod$(T)$ is a set of all models in which are all attribute implications from $T$ valid, i.e. we define it by

$$\text{Mod}(T) = \{(M_1, M_2) \mid ||A \stackrel{C}{\Rightarrow} B||_{(M_1,M_2)} = 1 \text{ for all } A \stackrel{C}{\Rightarrow} B \in T\}.$$

Let $M$ be set of models, i.e. $M \subseteq 2^{X_2} \times 2^{X_3}$. Then Fml$(M)$ is a set containing all attribute implications that are valid in all models in $M$, i.e.

$$\text{Fml}(M) = \{A \stackrel{C}{\Rightarrow} B \mid ||A \stackrel{C}{\Rightarrow} B||_{(M_1,M_2)} = 1 \text{ for all} (M_1, M_2) \in M\}.$$

**Remark 1.** It is easy to see that the mappings Fml and Mod form a Gallois connection between the set of all conditional attribute implications and the set of all models.

## Entailment

First, we define the notion of semantic entailment of conditional attribute implications.

**Definition 4.** Let $T$ be a theory. An implication $A \stackrel{C}{\Rightarrow} B$ semanticaly follows from $T$ iff it is valid in all models in Mod$(T)$, we denote this fact as $T \models A \stackrel{C}{\Rightarrow} B$

**Remark 2.** Clearly, $T \models A \stackrel{C}{\Rightarrow} B$ iff Mod$(T) \subseteq$ Mod$(\{A \stackrel{C}{\Rightarrow} B\})$.

In what follows we show, that the semantic consequence of conditional attribute implications can be axiomatized by a small set of derivation rules.

**Definition 5** (Amstrong axioms)**.** Define the following deduction rules:

$(Ax)\ \dfrac{}{A\cup B \stackrel{C}{\Rightarrow} B}$,

$(Cut)\ \dfrac{A \stackrel{C}{\Rightarrow} B,\ B\cup D \stackrel{C}{\Rightarrow} E}{A\cup D \stackrel{C}{\Rightarrow} E}$,

$(Spec)\ \dfrac{A \stackrel{C\cup D}{\Rightarrow} B}{A \stackrel{D}{\Rightarrow} B}$,

$(Ax)$ and $(Cut)$ are known to form the axiomatization of semantic consequence of dyadic attribute implications, $(Spec)$ is a new rule that handles the conditions.

**Definition 6.** A proof of $A \stackrel{C}{\Rightarrow} B$ from theory $T$ is a sequence of triadic attribute implications $A_i \stackrel{C_i}{\Rightarrow} B_i$, $i = 1, \dots, n$, such that: (1) $A_i \stackrel{C_i}{\Rightarrow} B_i$ either belongs to $T$ or can be derived from triadic attribute implications $A_j \stackrel{C_j}{\Rightarrow} B_j$, $j < i$, using (Ax), (Cut), (Spec). (2) $A \stackrel{C}{\Rightarrow} B$ is $A_n \stackrel{C_n}{\Rightarrow} B_n$. If there is a proof of $A \stackrel{C}{\Rightarrow} B$ from $T$ se say that $A \stackrel{C}{\Rightarrow} B$ syntactically follows from $T$ and denote it by $T \vdash A \stackrel{C}{\Rightarrow} B$.

**Theorem 1.** *(Ax), (Cut), and (Spec) are sound.*

**Definition 7.** We define the following rules:

$(Ref)\ \dfrac{}{A \stackrel{C}{\Rightarrow} A}$

$(Add)\ \dfrac{A \stackrel{C}{\Rightarrow} B,\ A \stackrel{C}{\Rightarrow} D}{A \stackrel{C}{\Rightarrow} B\cup D}$

$(Wea)\ \dfrac{A \stackrel{C}{\Rightarrow} B}{A\cup D \stackrel{C}{\Rightarrow} B}$

$(Pro)\ \dfrac{A \stackrel{C}{\Rightarrow} B\cup D}{A \stackrel{C}{\Rightarrow} B}$

$(Tra)\ \dfrac{A \stackrel{C}{\Rightarrow} B,\ B \stackrel{C}{\Rightarrow} D}{A \stackrel{C}{\Rightarrow} D}$

**Lemma 1.** *(Ref), (Add), (Wea), (Pro), (Tra) are sound.*

## Completeness

**Lemma 2.** *A theory $T$ is syntactically closed iff it is closed w.r.t. (Ax), (Cut), and (Spec).*

**Theorem 2.** *Let $T$ be a theory. If it is semantically closed then it is syntacticaly closed.*

**Theorem 3.** *Let $T$ be a theory. If it is syntactically closed, then it is semantically closed.*

**Theorem 4.** *Let $T$ be a theory. For any conditional attribute implication it holds*

$$T \vdash A \stackrel{C}{\Rightarrow} B \textit{ iff } T \models A \stackrel{C}{\Rightarrow} B.$$

### Attribute implications in triadic contexts

In this section we prove a theorem which yields a simple way of checking a validity of a conditional attribute implication in a triadic context.

**Definition 8.** Let $K = (X_1, X_2, X_3)$ be a triadic context. An implication $A \stackrel{C}{\Rightarrow} B$ is valid i $K$ iff for each $x_1 \in X_1$, all concepts of $\mathcal{B}(X_2, X_3, I^{23}_{\{x_1\}})$ are models of $A \stackrel{C}{\Rightarrow} B$. We denote it by $||A \stackrel{C}{\Rightarrow} B||_K = 1$.

**Theorem 5.** $||A \stackrel{C}{\Rightarrow} B||_K = 1$ *iff* $A^{(21C)} \subseteq B^{(21C)}$ *iff* $B \subseteq A^{(21C)(12C)}$.

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