T. Kühr, V. Vychodil: Confluence on Similarity and Generalized Pseudometric Spaces

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Introduction

The confluence is a crucial property of binary relations which are used to model substitutions in the theory of abstract rewriting systems. The classical notion of confluence of binary relations has already been the subject of extensive research. Besides their theoretical importance, notions related to rewriting were also applied in various fields of computer science, e.g. logic and functional programming, logical deductive systems, algebraic specification of abstract types. The basic motivation for the presented research is the fact that there are natural examples where the notion of substitutability is inherently fuzzy rather than crisp. Moreover, the universe of discourse is often equipped with a similarity relation or a metric providing an additional information which should be taken into account in the rewriting process.

Reduction and Confluence

Let R be a binary relation on a set X and assume that $\langle x, y\rangle \in R$ means that one may substitute y for x. The substitution "y for x" can be explained so that, whenever x does a certain job, y does it as well. An element $x \in X$ is called reducible if $\langle x, y\rangle \in R$ for some $y \in Y$; otherwise, x is called irreducible. By a reduction we mean any sequence $x_0, \ldots, x_n$ such that $\langle x_{i-1}, x_i\rangle \in R$ $(i = 1, \ldots, n)$. Relation R is called confluent whenever x is reducible to both y and y' then there is some z such that both y and y' are reducible to z.

Substitutability and Similarity Spaces

We elaborate on our previous results and extend the notions by equipping the universe by a similarity relation. Therefore, the prior definition of reducibility degree can be extended so that the similarity is taken into account. Namely, we can define a degree to which "there are z_1 and z_2 such that x is similar to z_1 and z_2 is substitutable for z_1 and z_2 is similar to y" as a more general degree of substitutability. Clearly, such an approach can yield more natural results then the original one.

Substitutability and Pseudometric Spaces

A analogous extension of substitutability is possible if one equips the universe by a generalized pseudometric specifying distances between elements. In this case, the notions related to substitutability should respect the given pseudometric by allowing "jumps" between close elements. Considering this type of distance-based similarity, we can say that x can be substituted by y if there are z_1 and z_2 such that x is near to z_1, y is near to z_2 and z_1 can be reduced to z_2 in the usual sense. In addition, it is natural to consider that the sum of distances of all "jumps" is delimited by a constant.

Generalized Confluence

We have introduced two generalizations of the notions of reducibility, convergence, divergence and confluence and investigate their mutual relationship. The first set of notions takes a fuzzy reduction and a similarity relation into consideration, the second one is based on a crisp reduction and a generalized pseudometric. The classical counterparts of the presented notions can be seen as a particular case of our definitions when the structure of truth degrees is the two-valued Boolean algebra and one uses a crisp equality, or a trivial metric, respectively.

Confluence on Similarity Spaces

In what follows, L denotes a complete residuated lattice, $\langle X, \approx\rangle$ is an $\mathbf{L}$-similarity space representing a universe X of all elements that can be used for substitution together with the indistinguishability relation $\approx$. Additionally, we consider a binary $\mathbf{L}$-relation $\rightharpoonup$ on X representing the substitutability relation.

Definition 1. The degrees of reducibility $\rightharpoonup^*_\approx$ are given by

$$x \rightharpoonup^*_\approx y = \bigvee_{\langle z_1, z_2, \ldots, z_{2k}\rangle \in X^{2\mathbb{N}_0}} (x \approx z_1 \otimes z_1 \rightharpoonup z_2 \otimes z_2 \approx z_3 \otimes \cdots \cdots \otimes z_{2k-1} \rightharpoonup z_{2k} \otimes z_{2k} \approx y),$$

where $x, y \in X$ and $X^{2\mathbb{N}_0} = \bigcup_{n \in \mathbb{N}_0} X^{2n}$.

Definition 2. The degrees of divergence $\downharpoonleft_\approx$ and convergence $\upharpoonleft_\approx$ are defined for all $x, y \in X$ as follows.

$$x \downharpoonleft_\approx y = \textstyle\bigvee_{z \in X} (x \rightharpoonup^*_\approx z \otimes y \rightharpoonup^*_\approx z)$$
$$x \upharpoonleft_\approx y = \textstyle\bigvee_{z \in X} (z \rightharpoonup^*_\approx x \otimes z \rightharpoonup^*_\approx y)$$

Definition 3. The degree $\mathrm{CFL}(\rightharpoonup)_\approx$ to which $\rightharpoonup$ is confluent with respect $\approx$ is defined by $\mathrm{CFL}(\rightharpoonup)_\approx = S(\upharpoonleft_\approx, \downharpoonleft_\approx)$.

Confluence Respecting a Pseudometric

In the following, we let $\langle X, \delta\rangle$ be a generalized pseudometric space and $\rightsquigarrow$ be a classical reduction relation on the universe set X.

Definition 4. An element $x \in X$ is said to be reducible to $y \in X$ by $\rightsquigarrow$ with the cumulative jump distance $d < \infty$ with respect to δ (briefly $x \rightsquigarrow^d_\delta y$) if there is a sequence of elements $z_1, \ldots, z_{2k} \in X$ such that $z_1 \rightsquigarrow z_2$ and $z_3 \rightsquigarrow z_4$ and …and $z_{2k-1} \rightsquigarrow z_{2k}$ and $\delta(x, z_1) + \delta(z_2, z_3) + \cdots + \delta(z_{2k}, y) = d$.

Definition 5. Elements $x, y \in X$ are said to be convergent with the cumulative jump distance d with respect to δ (shortly $x \downharpoonleft^d_\delta y$) if there is $z \in X$ such that $x \rightsquigarrow^{d_1}_\delta z$, $y \rightsquigarrow^{d_2}_\delta z$, and $d_1 + d_2 = d$. Analogously, $x, y \in X$ are said to be divergent with the cumulative jump distance d with respect to δ ($x \upharpoonleft^d_\delta y$) if there is $z \in X$ such that $z \rightsquigarrow^{d_1}_\delta x$, $z \rightsquigarrow^{d_2}_\delta y$, and $d_1 + d_2 = d$.

Definition 6. A relation $\rightsquigarrow$ is called confluent with respect to δ if for each $x, y \in X$, $x \upharpoonleft^{d_1}_\delta y$ implies $x \downharpoonleft^{d_2}_\delta y$ with $d_1 \geq d_2$.

Link between Generalizations

In this part of the poster, we let $\mathbf{L} = \langle [0,1], \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ be a complete residuated lattice defined on the real unit interval with $\otimes$ being a continuous Archimedean t-norm with a continuous additive generator f. Let us recall that a continuous additive generator f is a strictly decreasing continuous mapping $f\colon [0,1] \to [0, +\infty]$ with $f(1) = 0$ such that $a \otimes b = f^{(-1)}(f(a) + f(b))$ for all $a, b \in L$, where $f^{(-1)}$ denotes the pseudoinverse of f defined by $f^{(-1)}(x) = f^{-1}(x)$ if $x \leq f(0)$ and $f^{(-1)}(x) = 0$ otherwise.

Similarity and Generalized Pseudometric

Now, we can use the well-known connection between L-similarities and generalized pseudometrics to establish a relationship between the presented sets on notions. Recall that for given similarity $\approx$ on X, a mapping $\delta_\approx\colon X \times X \to [0, +\infty]$ defined by $\delta_\approx(x, y) = f(x \approx y)$ is a generalized pseudometric. Conversely, let δ be a generalized pseudometric on X. Then $\approx_\delta\colon X \times X \to [0, 1]$ defined by $(x \approx_\delta y) = f^{(-1)}(\delta(x, y))$ is a similarity on X.

From Pseudometric to Similarity

Let $\langle X, \delta\rangle$ be a generalized pseudometric space and denote by $\approx_\delta$ the similarity $\mathbf{L}$-relation corresponding to δ. This way, $\langle X, \approx_\delta\rangle$ is a similarity space corresponding to $\langle X, \delta\rangle$. Furthermore, $\rightharpoonup$ be an ordinary relation on X. As it is usual, we can consider $\rightharpoonup$ as a crisp $\mathbf{L}$-relation on X.

Theorem 7. *For any $x, y \in X$, $x \rightharpoonup^d_\delta y$ implies $x \rightharpoonup^*_{\approx_\delta} y \geq f^{(-1)}(d)$.*

Theorem 8. *Let $x, y \in X$ be arbitrary elements. Then, $x \upharpoonleft^d_\delta y$ implies $x \upharpoonleft_{\approx_\delta} y \geq f^{(-1)}(d)$ and $x \downharpoonleft^d_\delta y$ implies $x \downharpoonleft_{\approx_\delta} y \geq f^{(-1)}(d)$.*

Theorem 9. *If $\rightharpoonup$ is confluent with respect to δ then $\mathrm{CFL}(\rightharpoonup)_{\approx_\delta} = 1$.*

From Similarity to Pseudometric

Let $\langle X, \approx\rangle$ be a similarity space and $\langle X, \delta_\approx\rangle$ the corresponding generalized pseudometric space. Again, let $\rightharpoonup$ be an ordinary relation on X.

Theorem 10. *For arbitrary $x, y \in X$, $x \rightharpoonup^*_\approx y = a > 0$ implies that there is a nonincreasing sequence $\{d_n\}_{n=1}^\infty$ of distances such that $f(a) = \lim_{n\to\infty} d_n$ and $x \rightharpoonup^{d_n}_{\delta_\approx} y$ for each $n \in \mathbb{N}$.*

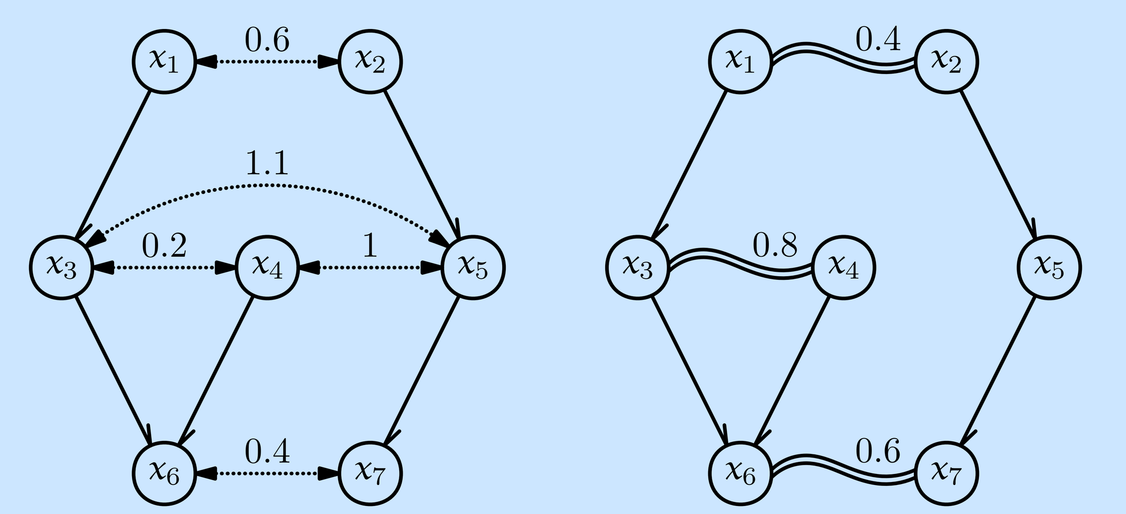
Theorem 11. *For any $x, y \in X$, if $x \upharpoonleft_\approx y = a > 0$ then there is a nonincreasing sequence of distances $\{d_n\}_{n=1}^\infty$ such that $f(a) = \lim_{n\to\infty} d_n$ and $x \upharpoonleft^{d_n}_{\delta_\approx} y$ for each $n \in \mathbb{N}$. Similarly for $\downharpoonleft_\approx$.*

Theorem 12. *Let $\mathrm{CFL}(\rightharpoonup)_\approx = 1$. If $x \upharpoonleft^d_{\delta_\approx} y$ and $d < f(0)$, then there is a nonincreasing sequence $\{d_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} d_n \leq d$ and $x \downharpoonleft^{d_n}_{\delta_\approx} y$ for each $n \in \mathbb{N}$.*

Example

Let $\langle X, \delta\rangle$ be a generalized pseudometric space and $\rightharpoonup$ a (crisp) binary relation which are depicted in the left part of the figure below. For sake of clarity, only distances with $\delta(x, y) < +\infty$ are included in the picture. Note, that $\rightharpoonup$ is confluent with respect to δ in the sense of Definition 6.

$\langle X, \delta\rangle, \rightharpoonup$ $\qquad$ $\langle X, \approx_\delta\rangle, \rightharpoonup$



Let $\mathbf{L}$ be a complete residuated lattice defined on the real unit interval with $\otimes$ being the Łukasiewicz t-norm. Note that $\otimes$ is continuous and Archimedean with a continuous additive generator $f(x) = 1 - x$. Using the presented results, one can transfer the background knowledge from $\langle X, \delta\rangle$ to $\langle X, \approx_\delta\rangle$ which is depicted in the right part of the picture. Again, truth degrees $x \approx y = 0$ are not included. By Theorem 9, we immediately get $\mathrm{CFL}(\rightharpoonup)_{\approx_\delta} = 1$ which can be verified by Definition 3.

Conclusion

We have introduced the notions related to substitutability (namely reducibility, convergence, divergence and confluence) induced by a fuzzy relation on a similarity space. The notions related to substitutability on generalized pseudometric spaces were also introduced and a connection between these notions and the notions on similarity spaces was established.

Future research

Topics that have not been considered include:

- issues of termination of (fuzzy) relations on similarity (pseudometric) spaces and further properties related to rewriting;
- preservation of properties like confluence by a-cuts;
- relationship to other types of compositions of $\mathbf{L}$-relations.

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