A Single Axiom for Boolean Algebras

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Eloy Renedo, Enric Trillias, Claudi Alsina (2003)

The only De Morgan algebras and the only orthomodular lattices in which

$$(x \wedge y^{\triangle})^{\triangle} = y \vee (x^{\triangle} \wedge y^{\triangle}) \text{ for all } x, y \in L. \quad (1)$$

holds, are Boolean algebras.

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An algebra $(L, \wedge, \vee, ^{\triangle})$ of type $(2, 2, 1)$ is a Boolean algebra iff

$$(L, \wedge, \vee) \text{ is a bounded lattice with } 1^{\triangle} = 0 \text{ and} \quad (2)$$

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An algebra $(L, \wedge, \vee, ^{\triangle})$ of type $(2, 2, 1)$ is a Boolean algebra iff

$$(L, \wedge, \vee) \text{ is a non empty lattice and} \tag{4}$$
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$$\implies x = (x \wedge y) \vee (x \wedge y^{\triangle}) \tag{7}$$
$$\implies x = (x \vee y) \wedge (x \vee y^{\triangle}) \tag{8}$$

Each concept x decomposes into a negative and a positive part with respect to any other concept y, as well for the disjunction as for the conjunction.

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Definition

A **weakly dicomplemented lattice** is a bounded lattice L equipped with two unary operations $^{\triangle}$ and $^{\nabla}$ called **weak complementation** and **dual weak complementation**, and satisfying for all $x, y \in L$

(1) $x^{\triangle\triangle} \leq x$,
(2) $x \leq y \implies x^{\triangle} \geq y^{\triangle}$,
(3) $(x \wedge y) \vee (x \wedge y^{\triangle}) = x$,

(1') $x^{\nabla\nabla} \geq x$,
(2') $x \leq y \implies x^{\nabla} \geq y^{\nabla}$,
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We call

- $x^{\triangle}$ the **weak complement** of x
- x^{∇} the **dual weak complement** of x
- $(x^{\triangle}, x^{\nabla})$ the **weak dicomplement** of x
- $(^{\triangle}, ^{\nabla})$ a **weak dicomplementation** on L
- $(L, \wedge, \vee, ^{\triangle}, 0, 1)$ a **weakly complemented lattice** and
- $(L, \wedge, \vee, ^{\nabla}, 0, 1)$ a **dual weakly complemented lattice**.

$(L, \wedge, \vee, ^{\triangle}, ^{\nabla}, 0, 1)$ wdl iff $(L, \wedge, \vee, 0, 1)$ bounded lattice and

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It follows

$y \vee y^{\triangle} = 1$,
$0^{\triangle} = 1$
$(x \wedge y)^{\triangle} = x^{\triangle} \vee y^{\triangle}$
$x^{\triangle\triangle\triangle} = x^{\triangle}$,
$x^{\nabla} \leq x^{\triangle}$

$y \wedge y^{\nabla} = 0$,
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$(x \vee y)^{\nabla} = x^{\nabla} \wedge y^{\nabla}$
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$x^{\triangle\nabla} \leq x \leq x^{\nabla\triangle}$

- $x \mapsto x^{\triangle\triangle}$ is a kernel operator on L
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Examples of wdl's

Boolean algebras

If $(B, \wedge, \vee, \bar{\ }, 0, 1)$ is a Boolean algebra then $(B, \wedge, \vee, \bar{\ }, \bar{\ }, 0, 1)$ is a weakly dicomplemented lattice.

Finite Lattices

Let L be a finite lattice. Set $J(L)$ the set of join irreducible and by $M(L)$ the set of meet irreducible elements of L respectively.

For $G \supseteq J(L)$ and $H \supseteq M(L)$, define $^{\triangle_G}$ and $^{\nabla_H}$ by

$$x^{\triangle_G} := \bigvee\{a \in G \mid a \not\leq x\} \quad \text{and} \quad x^{\nabla_H} := \bigwedge\{m \in H \mid m \not\geq x\}.$$

Thus $(L, \wedge, \vee, ^{\triangle_G}, ^{\nabla_H}, 0, 1)$ is a weakly dicomplemented lattice.

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Thus $(L, \wedge, \vee, {}^{\triangle_G}, {}^{\nabla_H}, 0, 1)$ is a weakly dicomplemented lattice.

Negation on closure and kernel systems

Let h be a closure operator on a set X and k a kernel operator on a set Y. For $A \subseteq X$ and $B \subseteq Y$ define $A^{\triangle_h} := h(X \setminus A)$ and $B^{\nabla_k} := k(Y \setminus B)$.

(i) $(h\mathcal{P}(X), \cap, \vee^h, {}^{\triangle_h}, h\emptyset, X)$, with $A_1 \vee^h A_2 := h(A_1 \cup A_2)$, is a weakly complemented lattice.

(i') $(k\mathcal{P}(Y), \wedge_k, \cup, {}^{\nabla_k}, \emptyset, kY)$, with $B_1 \wedge_k B_2 := k(B_1 \cap B_2)$, is a dual weakly complemented lattice.

(ii) If $h\mathcal{P}(X)$ is isomorphic to $k\mathcal{P}(Y)$, then h and k induce weakly dicomplemented lattice structures on $h\mathcal{P}(X)$ and on $k\mathcal{P}(Y)$ that are extensions of those in (i) and (i') above respectively.

Let φ be an isomorphism from $h\mathcal{P}(X)$ to $k\mathcal{P}(Y)$. Set $L := \{(x, y) \in h\mathcal{P}(X) \times k\mathcal{P}(Y) \mid y = \varphi(x)\}$. Then L has a weakly dicomplemented lattice structure induced by h and k.

Formal Concept Analysis

- is based on the the formalization of the notion of *concept*
- Traditional philosophers considered a **concept** to be determined by its extent and its intent. The **extent** consists of all objects belonging to the concept while the **intent** is the set of all attributes shared by all objects of the concept.
- The **concept hierarchy** states that a concept is more general if it contains more objects, or equivalently, if it is determined by less attributes.
- A **context** or universe of discourse can be seen as a relation involving objects and attributes of interest.
- What is the negation of a concept?
- How can it be formalized?

Formal Concept Analysis

- **formal context**: $\mathbb{K} := (G, M, \mathrm{I})$ with $\mathrm{I} \subseteq G \times M$.
- $G :\equiv$ set of **objects** $M :\equiv$ set of **attributes**.
- $g\,\mathrm{I}\,m :\iff (g, m) \in \mathrm{I}$. g has attribute m.

$$A' := \{m \in M \mid \forall g \in A\; g\,\mathrm{I}\,m\} \text{ and } B' := \{g \in G \mid \forall m \in B\; g\,\mathrm{I}\,m\}$$

- **Formal concept**: (A, B) with $A' = B$ and $B' = A$.
- A is the **extent** and B the **intent** of the concept (A, B).
- $\mathfrak{B}(\mathbb{K}) :=$ set of all formal concepts of $\mathbb{K}$.
- A concept (A, B) is a **subconcept** of a concept (C, D) if $A \subseteq C$ (or equivalently, $D \subseteq B$). write $(A, B) \leq (C, D)$.
- $c : X \mapsto X''$ is a closure operator on $\mathcal{P}(G)$ and on $\mathcal{P}(M)$.
- $(\mathfrak{B}(\mathbb{K}); \leq)$ is a complete lattice, called **concept lattice** of $\mathbb{K}$.

Weak Negation and weak opposition

To formalize a negation two operations are introduced:

Definition

Let $\mathbb{K}$ be a context and (A, B) a formal concept of $\mathbb{K}$. We define

its **weak negation** by $\quad (A, B)^{\triangle} := ((G \setminus A)'', (G \setminus A)')$

and its **weak opposition** by $\quad (A, B)^{\nabla} := ((M \setminus B)', (M \setminus B)'')$.

$\mathfrak{A}(\mathbb{K}) := (\mathfrak{B}(\mathbb{K}); \wedge, \vee, ^{\triangle}, ^{\nabla}, 0, 1)$ is called the **concept algebra** of the formal context $\mathbb{K}$, where $\wedge$ and $\vee$ denote the meet and the join operations of the concept lattice.

$\mathfrak{A}(\mathbb{K}) := (\mathfrak{B}(\mathbb{K}); \wedge, \vee, ^{\triangle}, ^{\nabla}, 0, 1)$ is a weakly dicomplemented lattice.

Boundness for free

Weakly complemented lattice are exactly non empty lattices satisfying the equations (1)–(3).
i.e. Each nonempty lattice satisfying (1)-(3) is bounded.

Let L be a non empty lattice satisfying (1)–(3').

- Let $x \in L$. Set $1 := x \vee x^{\triangle}$ and $0 := 1^{\triangle}$. Let $y \in L$ (arbitrary).
- $1 \geq y \wedge 1 = y \wedge (x \vee x^{\triangle}) \geq (y \wedge x) \vee (y \wedge x^{\triangle}) = y$
- Thus $x \vee x^{\triangle}$ is the greatest element of L.
- $(y \wedge y^{\triangle})^{\triangle} \geq y^{\triangle} \vee y^{\triangle\triangle} = 1; \implies (y \wedge y^{\triangle})^{\triangle} = 1$.
- Let $z \in L$

$$0 \wedge z = 1^{\triangle} \wedge z = (y \wedge y^{\triangle})^{\triangle\triangle} \wedge z \leq y \wedge y^{\triangle} \wedge z \leq y \wedge z$$

$$0 \wedge z^{\triangle} = 1^{\triangle} \wedge z^{\triangle} = (y \wedge y^{\triangle})^{\triangle\triangle} \wedge z^{\triangle} \leq y \wedge y^{\triangle} \wedge z^{\triangle} \leq y \wedge z^{\triangle}$$

$$0 = (0 \wedge z) \vee (0 \wedge z^{\triangle}) \leq (y \wedge z) \vee (y \wedge z^{\triangle}) = y$$

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$$0 \wedge z = 1^{\triangle} \wedge z = (y \wedge y^{\triangle})^{\triangle\triangle} \wedge z \leq y \wedge y^{\triangle} \wedge z \leq y \wedge z$$

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Weakly Dicomplemented Lattices with Negation

A weakly dicomplemented lattice is said to be **with negation** if $^{\triangle} = ^{\nabla}$.

Theorem

A weakly dicomplemented lattice $(L, \wedge, \vee, ^{\triangle}, ^{\nabla}, 0, 1)$ is with negation iff $(L, \wedge, \vee, ^{\triangle}, 0, 1)$ and $(L, \wedge, \vee, ^{\nabla}, 0, 1)$ are Boolean algebras.

- Let $(L, \wedge, \vee, ^{\triangle}, ^{\nabla}, 0, 1)$ with $^{\triangle} = ^{\nabla}$.
- $x \vee x^{\triangle} = 1$ and $x \wedge x^{\nabla} = 0$. Then $x^{\triangle}$ is a complement of x.
- What about the distributivity?
- The idea is to show that any weakly dicomplemented lattice with negation having at least three elements is not subdirectly irreducible.
- i.e. for any $L \in \mathrm{WDN}$ with $|L| \geq 3$ there is $\theta_1, \theta_2 \in \mathrm{Con}(L)$ such that $\theta_1 \cap \theta_2 = \Delta$, the trivial congruence.

Weakly Dicomplemented Lattices with Negation

A weakly dicomplemented lattice is said to be **with negation** if $^{\triangle} = ^{\nabla}$.

Theorem

A weakly dicomplemented lattice $(L, \wedge, \vee, ^{\triangle}, ^{\nabla}, 0, 1)$ *is with negation iff* $(L, \wedge, \vee, ^{\triangle}, 0, 1)$ *and* $(L, \wedge, \vee, ^{\nabla}, 0, 1)$ *are Boolean algebras.*

- Let $(L, \wedge, \vee, ^{\triangle}, ^{\nabla}, 0, 1)$ with $^{\triangle} = ^{\nabla}$.
- $x \vee x^{\triangle} = 1$ and $x \wedge x^{\nabla} = 0$. Then $x^{\triangle}$ is a complement of x.
- What about the distributivity?
- The idea is to show that any weakly dicomplemented lattice with negation having at least three elements is not subdirectly irreducible.
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(i) For $c \in L \setminus \{0,1\}$, we have $[c,1] \cong [0,c^{\triangle}]$, since

$$u_{c^{\triangle}} : [c,1] \to [0,c^{\triangle}], \quad x \mapsto x \wedge c^{\triangle} \qquad \text{and} \qquad v_c : [0,c^{\triangle}] \to [c,1], \quad x \mapsto x \vee c$$

are order preserving and inverse of each other.

(ii) The maps

$$f_1 : L \to [c^{\triangle},1] \to [0,c^{\triangle\triangle}] = [0,c], \quad x \mapsto x \vee c^{\triangle} \mapsto (x \vee c^{\triangle}) \wedge c = x \wedge c$$

and

$$f_2 : L \to [c,1] \to [0,c^{\triangle}], \quad x \mapsto x \vee c \mapsto (x \vee c) \wedge c^{\triangle} = x \wedge c^{\triangle}$$

are lattice homomorphisms.

(iii) We set $\theta_1 := \ker f_1$ and $\theta_2 := \ker f_2$. Then $\theta_1 \cap \theta_2 = \Delta$.

$$\begin{aligned}(x,y) \in \theta_1 \cap \theta_2 &\implies x \wedge c = y \wedge c \text{ and } x \wedge c^{\triangle} = y \wedge c^{\triangle} \\ &\implies x = (x \wedge c) \vee (x \wedge c^{\triangle}) = (y \wedge c) \vee (y \wedge c^{\triangle}) = y \\ &\implies (x,y) \in \Delta.\end{aligned}$$

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Corollary

$(L, \wedge, \vee, ^{\triangle}, 0, 1)$ is a Boolean algebra iff $(L, \wedge, \vee)$ is a non empty lattice in which

1. $x^{\triangle\triangle} = x$,
2. $x \leq y \implies x^{\triangle} \geq y^{\triangle}$
3. $(x \wedge y) \vee (x \wedge y^{\triangle}) = x = (x \vee y) \wedge (x \vee y^{\triangle})$.

hold.

Recall that

(1) $x^{\triangle\triangle} \leq x$,

(2) $x \leq y \implies x^{\triangle} \geq y^{\triangle}$,

(3) $(x \wedge y) \vee (x \wedge y^{\triangle}) = x$,

(1') $x^{\nabla\nabla} \geq x$,

(2') $x \leq y \implies x^{\nabla} \geq y^{\nabla}$,

(3') $(x \vee y) \wedge (x \vee y^{\nabla}) = x$.

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Corollary

$(L,\wedge,\vee,^{\triangle},0,1)$ is a Boolean algebra iff $(L,\wedge,\vee)$ is a non empty lattice in which

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(2') $x \leq y \implies x^{\nabla} \geq y^{\nabla}$,

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Theorem (A new axiom for Boolean algebras)

An algebra $(L, \wedge, \vee, ^{\triangle}, 0, 1)$ *is a Boolean algebra iff* $(L, \wedge, \vee)$ *is a non empty lattice in which*

$$(x \wedge y) \vee (x \wedge y^{\triangle}) = (x \vee y) \wedge (x \vee y^{\triangle}) \text{ holds for all } x, y \in L \qquad (\ddagger).$$

(i) $x \geq (x \wedge y) \vee (x \wedge y^{\triangle}) = (x \vee y) \wedge (x \vee y^{\triangle}) \geq x$ implies $(x \wedge y) \vee (x \wedge y^{\triangle}) = x = (x \vee y) \wedge (x \vee y^{\triangle})$.

(ii) $x = (x \vee y) \wedge (x \vee y^{\triangle}); \implies y \wedge y^{\triangle} = 0;$

$x = (x \wedge x^{\triangle}) \vee (x \wedge x^{\triangle\triangle}) = 0 \vee (x \wedge x^{\triangle\triangle}) = x \wedge x^{\triangle\triangle}$

Hence $x \leq x^{\triangle\triangle}$.

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Hence $x \geq x^{\triangle\triangle}$. Therefore $x = x^{\triangle\triangle}$.

(iii) Let $x \leq y$.

$x \vee x^{\triangle} = 1 \implies y \vee x^{\triangle} = 1$. Thus

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$x = (x \vee x^{\triangle}) \wedge (x \vee x^{\triangle\triangle}) = 1 \wedge (x \vee x^{\triangle\triangle}) = x \vee x^{\triangle\triangle}$

Hence $x \geq x^{\triangle\triangle}$. Therefore $x = x^{\triangle\triangle}$.

(iii) Let $x \leq y$.

$x \vee x^{\triangle} = 1 \implies y \vee x^{\triangle} = 1$. Thus

$x^{\triangle} = (x^{\triangle} \vee y^{\triangle}) \wedge (x^{\triangle} \vee y^{\triangle\triangle}) = (x^{\triangle} \vee y^{\triangle}) \wedge (x^{\triangle} \vee y) = x^{\triangle} \vee y^{\triangle}$

and $x^{\triangle} \geq y^{\triangle}$.

(i) $x \geq (x \wedge y) \vee (x \wedge y^{\triangle}) = (x \vee y) \wedge (x \vee y^{\triangle}) \geq x$ implies $(x \wedge y) \vee (x \wedge y^{\triangle}) = x = (x \vee y) \wedge (x \vee y^{\triangle})$.

(ii) $x = (x \vee y) \wedge (x \vee y^{\triangle}); \implies y \wedge y^{\triangle} = 0;$
$x = (x \wedge x^{\triangle}) \vee (x \wedge x^{\triangle\triangle}) = 0 \vee (x \wedge x^{\triangle\triangle}) = x \wedge x^{\triangle\triangle}$
Hence $x \leq x^{\triangle\triangle}$.
$x = (x \wedge y) \vee (x \wedge y^{\triangle}); \implies y \vee y^{\triangle} = 1;$
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and $x^{\triangle} \geq y^{\triangle}$.