Valuations on Finite Lattices

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joint work with Stefan Schmidt
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Motivation

Data Analysis

- Qualitative Data Analysis
  - Formal Concept Analysis
  - Rough Set Theory
  - ...are based on closure operators.
- Quantitative Data Analysis
  - Statistics
  - Data Mining
  - ...are based on measure theory.
- Aim: Bridging qualitative and quantitative Data Analysis
- Tools: Closure operators $\longleftrightarrow$ measures

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- The carrier set $\mathcal{S}$ is a σ-**algebra**,
  i.e a collection $\mathcal{S}$ of subsets closed under complementation and countable unions, i.e a special Boolean algebra.

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We need distributivity here.!

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- An isotone measure with $\mu(\emptyset) = 0$ and $\mu(\bigcup \mathcal{S}) = 1$ (**fuzzy measure**) is
  - **modular** if $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B), \ \forall A, B \in \mathcal{S}$,
  - **submodular** if $\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B), \ \forall A, B \in \mathcal{S}$ and
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Let $L \neq \emptyset$ be a finite lattice. An **evaluation** on L is a map $r : L \to \mathbb{R}$. It is

- **modular** if $r(x \vee y) + r(x \wedge y) = r(x) + r(y)$,
- **submodular** if $r(x \vee y) + r(x \wedge y) \leq r(x) + r(y)$,
- **supermodular** if $r(x \vee y) + r(x \wedge y) \geq r(x) + r(y)$,
- a **valuation** if r is isotone and modular.

Notations and Examples

- L is a finite on-empty lattice.
- $x \lessdot y :\iff x < y$ and $x < a \leq y \implies a = y$ i.e. y covers x
- $a \neq 0$ is **join irreducible** $:\iff \bigvee\{x \in L \mid x < a\} \lessdot a$. In this case $a_* := \bigvee\{x \in L \mid x < a\}$ is the unique lower neighbour of a.
- $J(L) :=$ the set of join irreducible elements of L.
- $a \neq 1$ is **meet irreducible** $:\iff a \lessdot \bigwedge\{x \in L \mid x < a\}$. In this case $a^* := \bigwedge\{x \in L \mid x > a\}$ is the unique upper neighbour of a.
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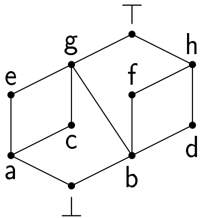

- $J(L) = \{a, b, c, d, e, f\}$. $M(L) = \{c, d, e, f, g, h\}$
- $a_* = \bot$, $f_* = b$, $f^* = h$, $h^* = \top$
- $\mathrm{h}(P) :=$ length of P and $\mathrm{h}_L(x) := \mathrm{h}([\bot, x])$
- $\mathrm{d}_L(x) := \mathrm{h}([x, \top])$ and $\mathrm{h}^L(x) := \mathrm{h}(L) - \mathrm{d}_L(x)$
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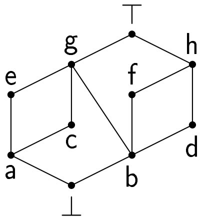

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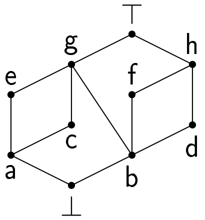

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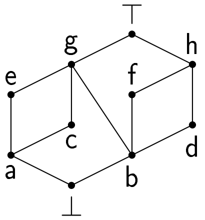

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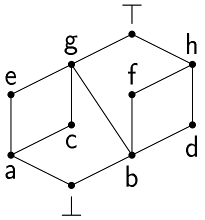

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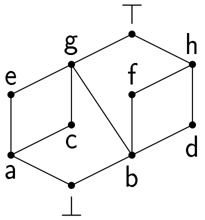

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Some issues to be addressed

1. How many modular evaluations can we define on a finite lattice?
2. When is h_L modular, supermodular or submodular?
3. When is h^L modular, supermodular or submodular?
4. When are h_L and h^L both equal?
5. $\mathrm{h}_L(x) + \mathrm{d}_L(x) \leq \mathrm{h}(L)$. When do we have equality?

Notations

- $\mathbb{R}^L := \{r : L \to \mathbb{R}\}$ is a real vector space (of evaluations on L).
- $\mathrm{Mod}(L, \mathbb{R}) :=$ set of modular evaluations on L
- $\mathrm{Mod}_0(L, \mathbb{R}) := \{r \in \mathrm{Mod}(L, \mathbb{R}) \mid r(\bot) = 0\}$ (normalized mod. eval.)
- $\mathrm{Mod}_0(L, \mathbb{R}) \leq \mathrm{Mod}(L, \mathbb{R}) \leq \mathbb{R}^L$ (as subspaces)
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Modular dimension of finite distributive lattices

- L is distributive $:\iff x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
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  $:\iff M_3 \not\leq L$ and $N_5 \not\leq L$.
- $\mathrm{md}_0(L) := \dim \mathrm{Mod}_0(L, \mathbb{R})$ (normalized modular dimension) of L.
- $r \in \mathrm{Mod}_0(L, \mathbb{R}) \implies r(x \vee y) = r(x) + r(y) - r(x \wedge y)$ and $r(0) = 0$.
- Denote by $r|_{J(L)}$ the resctriction of r on $J(L)$. Is there any characterization of such $r|_{J(L)}$? Can any $s : J(L) \to \mathbb{R}$ be extended to a modular evaluation on L.
- $U \subseteq J(L)$ is **modfree** if any $r : U \to \mathbb{R}$ extends $\tilde{r} \in \mathrm{Mod}(L)$.
- A maximal modfree subset of $J(L)$ is called a **modbasis**.
- Finding a maximal modfree subset of L is equivalent to finding the normalized modular dimension of L.

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The normalized modular dimension of a finite lattice L is equal to the number of its join irreducible elements if and only if L is distributive.

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Theorem

The normalized modular dimension of a finite lattice L is equal to the number of its join irreducible elements if and only if L is distributive.

Modular dimension of finite distributive lattices

- L is distributive $:\iff x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
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Upper bound of $\mathrm{md}(L)$

Let L be a finite lattice. Then $\mathrm{md}_0(L) \leq |J(L)|$.

- The application $\begin{array}{rccc} \psi & : & \mathrm{Mod}_0(L, \mathbb{R}) & \to & \mathbb{R}^{J(L)} \\ & & r & \mapsto & r|_{J(L)} \end{array}$ is an injective and linear map. Then,
- If $r_1|_{J(L)} = r_2|_{J(L)}$ and $x \in L \setminus J(L)$ is minimal wrt $r_1(x) = r_2(x)$ is not yet proved, there are $s, t < x$ such that $s \vee t = x$.

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Corollary

Let L be a finite lattice an r an evaluation on L. Then

1. $r \in \mathrm{Mod}_0(L, \mathbb{R})$ and $r|_{J([0,x])} = \bar{0}$ implies $r|_{[0,x]} = \bar{0}$.
2. For $p \in J(L)$, we define $\delta_p^{J(L)} \in \{0,1\}^{J(L)}$ by

$$\delta_p^{J(L)}(x) = 1 \iff x = p$$

If $r_p \in \mathrm{Mod}_0(L, \mathbb{R})$ and $r_p|_{J(L)} = \delta_p^{J(L)}$ then $p \not\leq x \implies r_p(x) = 0$.

Modular dimension of a product

Let L_1 and L_2 be finite lattices. Then

$$\mathrm{md}_0(L_1 \times L_2) = \mathrm{md}_0(L_1) + \mathrm{md}_0(L_2).$$

- For $r \in \mathrm{Mod}_0(L_1 \times L_2, \mathbb{R})$ set $r_1(x) := r(x,0)$ and $r_2(y) := r(0,y)$.
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Let L be a finite lattice. If $\mathrm{md}_0(L) = |J(L)|$ then L is distributive.

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Upper bound reached $\implies$ Distributivity

Let L be a finite lattice. If $\mathrm{md}_0(L) = |J(L)|$ then L is distributive.

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Constructing modular evaluations on distributive lattices

Let L be a distributive lattice.

- For $r \in \mathbb{R}^{J(L)}$ the $\begin{array}{rcl} \tilde{r} : L & \to & \mathbb{R} \\ x & \mapsto & \sum\limits_{\substack{a \leq x \\ a \in J(L)}} r(a) \end{array}$ is a modular evaluation.
- For $r \in \mathrm{Mod}(L, \mathbb{R})$, $a \in J(L)$ and $\lambda \in \mathbb{R}$ the function $r^{\uparrow a+\lambda}$ defined by $\begin{array}{rcl} r^{\uparrow a+\lambda} : L & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} r(x) & \text{if } a \not\leq x \\ r(x) + \lambda & \text{if } a \leq x \end{cases} \end{array}$ is modular.
- Any $r \in \mathbb{R}^{J(L)}$ extends to a modular evaluation $\hat{r} \in \mathrm{Mod}_0(L, \mathbb{R})$.

$\mathrm{md}_0(L) = |J(L)| \iff L$ is distributive.

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$$= \tilde{r}(x) + \sum_{\substack{a \leq y \\ a \nleq x \\ a \in J(L)}} r(a) + \sum_{\substack{a \leq x \wedge y \\ a \in J(L)}} r(a)$$

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Open Problem

Similar results for modular lattices?

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Closure and kernel operators on a poset $(P, \leq)$

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$r : L \to \mathbb{R}$ is submodular iff $r(x \vee y) + r(x \wedge y) \leq r(x) + r(y)$.

From submodular evaluations to closure operators

Let r be a submodular and isotone evaluation on L. Then

(i) For each $x \in L$, the set $\{y \in L \mid y \geq x \text{ and } r(y) = r(x)\}$ has a greatest element (denoted $c_r(x)$).

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- $$\begin{aligned} x \leq \mathrm{c}_r(y) &\implies x \leq \mathrm{c}_r(x) \wedge \mathrm{c}_r(y) \leq \mathrm{c}_r(x) \\ &\implies r(x) = r\left(\mathrm{c}_r(x) \wedge \mathrm{c}_r(y)\right) = r\left(\mathrm{c}_r(x)\right) \\ &\implies r(x) + r\left(\mathrm{c}_r(x) \vee \mathrm{c}_r(y)\right) \leq r(x) + r\left(\mathrm{c}_r(y)\right) \\ &\implies r\left(\mathrm{c}_r(x) \vee \mathrm{c}_r(y)\right) = r\left(\mathrm{c}_r(y)\right) = r(y) \\ &\implies \mathrm{c}_r(x) \vee \mathrm{c}_r(y) \leq \mathrm{c}_r(y) \text{ and } \mathrm{c}_r(x) \leq \mathrm{c}_r(y). \end{aligned}$$

Bridging quantitative and qualitative data analysis

From closure operators back to submodular evaluations

1. Let c be a closure operator on L. If $r : c(L) \to \mathbb{R}$ is strict isotone and submodular then $r \circ c$ is an isotone submodular evaluation on L.
2. The closure operator $\mathrm{c}_{r\circ c}$ generated by $r \circ c$ is equal to c.

1. $r \circ c$ is isotone. Let $x, y \in L$.

$$\begin{aligned} r \circ c(x \vee y) + r \circ c(x \wedge y) &= r(c(x) \vee_{cL} c(y)) + r(c(x \wedge y)) \\ &\leq r(c(x) \vee_{cL} c(y)) + r(c(x) \wedge c(y)) \\ &\leq r(c(x)) + r(c(y)) \\ &\leq r \circ c(x) + r \circ c(y) \end{aligned}$$

2. $$\begin{aligned} \mathrm{c}_{r\circ c}(x) &:= \max\{y \in L \mid y \geq x \text{ and } r \circ c(y) = r \circ c(x)\} \\ &= \max\{y \in L \mid y \geq x \text{ and } c(y) = c(x)\} \\ &= c(x) \end{aligned}$$

Bridging quantitative and qualitative data analysis

We still need a strict isotone and submodular evaluation $r : c(L) \to \mathbb{R}$

The function $\begin{array}{rcl} s : L & \to & \mathbb{R} \\ x & \mapsto & 2^{\mathrm{h}(L)} - 2^{\mathrm{d}_L(x)} \end{array}$ is strict isotone and submodular.

Closure, kernel operators $\leftrightarrow$ (isotone) sub-, supermodular evaluations

(i) For each submodular evaluation $r : L \to \mathbb{R}$ there is a closure operator c_r on L such that $r = r \circ \mathrm{c}_r$.

(ii) For each supermodular evaluation $r : L \to \mathbb{R}$ there is a kernel operator k_r on L such that $r = r \circ \mathrm{k}_r$.

(i') For each closure operator c on L, there is a submodular evaluation $r_c : L \to \mathbb{R}$ such that $c = \mathrm{c}_{r_c}$.

(ii') For each kernel operator k on L, there is a supermodular evaluation $r_k : L \to \mathbb{R}$ such that $k = \mathrm{k}_{r_k}$

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Formal Concept Analysis

- started in the eighties by Rudolf Wille,
- has established himself as own research field
- has been successfully used for conceptual clustering and rule generation, for Web mining, etc ...
- is based on the the formalization of the notion of *concept*
- Traditional philosophers considered a **concept** to be determined by its extent and its intent. The **extent** consists of all objects belonging to the concept while the **intent** is the set of all attributes shared by all objects of the concept.
- The **concept hierarchy** states that a concept is more general if it contains more objects, or equivalently, if it is determined by less attributes.
- A **context** or universe of discourse can be seen as a relation involving objects and attributes of interest.

Formal Concept Analysis

- A **formal context** is a triple $\mathbb{K} := (G, M, \mathrm{I})$ such that $\mathrm{I} \subseteq G \times M$.
- $G :\equiv$ set of **objects** $\qquad M :\equiv$ set of **attributes**.
- $g \, \mathrm{I} \, m :\iff (g, m) \in \mathrm{I}$. $\qquad g$ has attribute m.

$$A' := \{m \in M \mid \forall g \in A \; g \, \mathrm{I} \, m\} \quad \text{and} \quad B' := \{g \in G \mid \forall m \in B \; g \, \mathrm{I} \, m\}.$$

- A **formal concept** of $\mathbb{K}$ is a pair (A, B) with $A' = B$ and $B' = A$.
- A is the **extent** and B the **intent** of the concept (A, B).
- $\mathfrak{B}(\mathbb{K}) :=$ set of all formal concepts of $\mathbb{K}$.
- A concept (A, B) is a **subconcept** of a concept (C, D) if $A \subseteq C$ (or equivalently, $D \subseteq B$). $\qquad$ write $(A, B) \leq (C, D)$.
- $c : X \mapsto X''$ is a closure operator on $\mathcal{P}(G)$ and on $\mathcal{P}(M)$.
- $\mathrm{Ext}(\mathbb{K}) := c(\mathcal{P}(G)) \cong^d c(\mathcal{P}(M)) =: \mathrm{Int}(\mathbb{K})$.
- $(\mathfrak{B}(\mathbb{K}); \leq)$ is a complete lattice, called **concept lattice** of $\mathbb{K}$.

FCA: Reconstructing closures from measures on G or M

- Let $\mathbb{K} := (G, M, \mathrm{I})$ be a formal context and $\mathrm{c}_{\mathbb{K}} : 2^G \to 2^G,\ X \mapsto X''$ the corresponding a closure operator on 2^G.
- If P_G is a probability measure on $\mathrm{Ext}(\mathbb{K}) = \mathrm{c}_{\mathbb{K}}(2^G)$ then $P_G \circ \mathrm{c}_{\mathbb{K}}$ is an isotone submodular function on 2^G
- If P_G is strict isotone on $\mathrm{Ext}(\mathbb{K})$, for example $P_G(X) = \frac{|X|}{|G|}$ (uniform) then $\mathrm{c}_{\mathbb{K}}$ can be "reconstructed" from $P_G \circ \mathrm{c}_{\mathbb{K}}$.
- For $X \subseteq G,\ \mathrm{c}_{\mathbb{K}}(X) = \max\{A \mid X \subseteq A \subseteq G \text{ and } P_G(A) = P_G(X)\}$
- $\begin{array}{rcl} f : \mathfrak{B}(\mathbb{K}) & \to & \mathbb{R} \\ (A, B) & \mapsto & P_G(A) \end{array}$ is isotone submodular on $\mathfrak{B}(\mathbb{K})$.

- For $N \subseteq M$, the map $\begin{array}{rcl} c_N : \mathfrak{B}(\mathbb{K}) & \to & \mathfrak{B}(\mathbb{K}) \\ (A, B) & \mapsto & ((B \cap N)', (B \cap N)'') \end{array}$ is a closure operator on $\mathfrak{B}(\mathbb{K})$.
- Let P_M be probability measure on 2^M. Then $1 - P_M$ is submodular on $\mathrm{Int}(\mathbb{K})^d$ and $\begin{array}{rcl} g : \mathfrak{B}(\mathbb{K}) & \to & \mathbb{R} \\ (A, B) & \mapsto & 1 - P_M((B \cap N)'') \end{array}$ isotone and submodular on $\mathfrak{B}(\mathbb{K})$.
- If P_M is uniform, then c_N can be reconstructed from g.