Possibilistic logic
1. Basics
2. Applications
3. Extensions
4. Possibility theory / FCA

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1. Basics

- What's about
- Background on possibility theory
- Degree of uncertainty vs. degree of truth
- Standard possibilistic logic (syntax and semantics)
- Inconsistency handling
- Guaranteed possibility-based logic

What's about?

- (p, **a**)
- p : classical logic formula
- **a** : level in a scale (0,1]
- N(p) ≥ a   N : necessity measure
- N(p) ≥ a, N(¬p ∨ q) ≥ **b** |− N(q) ≥ min(**a**, **b**) (Prade, 1982)
- Theophrastus
- Nicholas Rescher (Plausible Reasoning,1976)

Developed with many co-authors

Including

- Didier Dubois
- Jérôme Lang
- Salem Benferhat
- Souhila Kaci
- Steven Schockaert
- …

can be used for modeling

- uncertainty

(p, **a**) p is true with certainty **a**

- preferences

(p, **a**) making p true is a goal with priority **a**

BOOLEAN POSSIBILITY THEORY

Set-functions acting as measures of uncertainty

If all we know is that $\boldsymbol{x} \in \boldsymbol{E}$ *then*

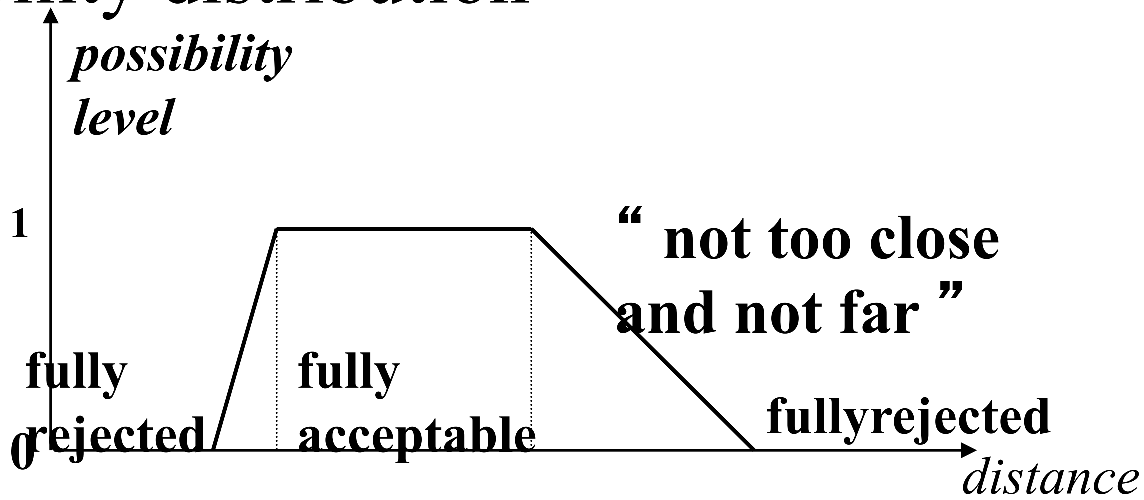
- Event A is possible if $A \cap E \neq \emptyset$ (logical consistency)

  $\Pi(A) = 1$, and 0 otherwise
- Event A is sure if $E \subseteq A$ (logical deduction)

  $N(A) = 1$, and 0 otherwise
- **Axiom** : $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$;
- Axiom : $N(A \cap B) = \min(N(A), N(B))$.
- ***Close to a simple modal logic (KD45)***

Possibility theory - 1

- **π** a possibility distribution



- π(u) = 0 means X = u is totally excluded
- π(u) = 1 means X = u is completely possible
- *Possibility measure*
- Π**(A) = sup{π(u) : u ∈ A}** to what extent event A is *consistent* with the information X is F
- $\Pi(\mathbf{A} \cup \mathbf{B}) = \mathbf{max}(\Pi(\mathbf{A}), \Pi(\mathbf{B}))$

Possibility theory - 2

- L. A. Zadeh, 1978
- G. L. S. Shackle, 1949-1962 *degree of surprise*
- also

  L. J. Cohen, D. Lewis, Grove, Maslov, Shilkret, …

For Zadeh: linguistic terms → possibility distribution

Peter is *young*

Here possibility distribution defined on *a set of interpretations* induced by a logical language

Possibility theory - 3

- $N(\mathbf{A}) = \mathbf{1} - \Pi(\mathbf{A^c})$

  $= 1 - \sup\{\pi(u) : u \notin A\}$

to what extent event A is *implied* by the information

- **N(A) = 1**:

  A is certain (true in all non-impossible situations)

- **N(A) > 0**: Given that X is F, A is normally true (true in all the most plausible situations)

- $\mathbf{N(A \cap B) = min(N(A), N(B))}$

Degree of uncertainty *vs.* degree of truth

- **$\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$**
- **$\Pi(A \cap B) \leq \min(\Pi(A), \Pi(B))$!**

- **$N(A \cap B) = \min(N(A), N(B))$**
- **$N(A \cup B) \geq \max(N(A), N(B))$!**

- **Π**, **N** are *increasing* wrt *set inclusion*

fuzzy measure, capacity

- Degree of uncertainty *cannot be decomposable wrt to all logical connectives*

degrees of uncertainty pertain to classical formulas

Boolean lattice

- Degree of truth *may be decomposable wrt to all logical connectives*

degrees of truth pertain to non classical formulas

distributive lattice

Possibility theory - 4

- Modeling ignorance

$$\Pi(A \cup A^c) = \max(\Pi(A), \Pi(A^c))$$

$$N(A \cap A^c) = \min(N(A), N(A^c))$$

- **Qualitative** possibility theory

  vs. **Quantitative** possibility theory

$$\Pi(A \cap B) = \Pi(A \mid B) * \Pi(B)$$

  with $* = \min$ or $* = \times$

- Bayesian possibilistic network

Possibility theory - 5

- $\Pi(A) = 1$ **and** $N(A) = 1$ A certainly true
- $\Pi(A) = 1$ **and** $N(A) > 0$ A true somewhat certain
- $\Pi(A) = 1$ **and** $N(A) = 0$ total ignorance about A
- $\Pi(A) < 1$ **and** $N(A) = 0$ A false somewhat certain
- $\Pi(A) = 0$ **and** $N(A) = 0$ A certainly false

Possibility theory - 6

- *Guaranteed (strong) possibility measure*
- **$\Delta(A) = \inf\{ \pi(u) : u \in A\}$**
- to what extent **all** situations where A is true are possible for sure
- **$\Delta(A \cup B) = \min(\Delta(A), \Delta(B))$**
- decreasing w. r. t. set inclusion
- $\nabla(A) = 1 - \Delta(A^c)$ **(weak necessity)**
- **$\nabla(A \cap B) = \max(\nabla(A), \nabla(B))$**

Possibility theory - 7

- **$\Delta(A \cup B) = \min(\Delta(A), \Delta(B))$**
- **$\Delta(A \cap B) \geq \max(\Delta(A), \Delta(B))$**

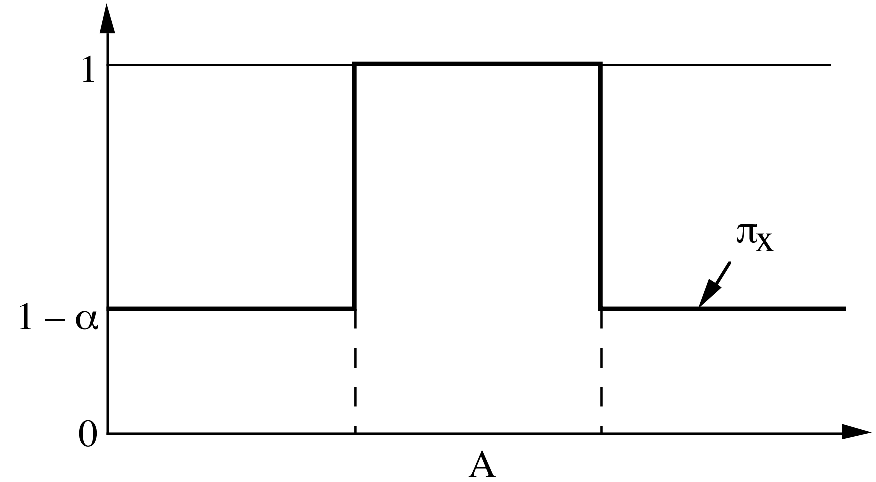
- **$\nabla(A \cap B) = \max(\nabla(A), \nabla(B))$**
- **$\nabla(A \cup B) \leq \min(\nabla(A), \nabla(B))$**

- Δ, ∇ are **decreasing** set functions

Possibility theory - 8

- $\Pi(\mathbf{A})$ *max* over **A**
- $N(\mathbf{A})$ 1 - *max* over $\mathbf{A}^c$
- $\Delta(\mathbf{A})$ *min* over **A**
- $\nabla(\mathbf{A})$ 1 - *min* over $\mathbf{A}^c$

Certainty-qualification



- Attaching a degree of certainty α to event A
- It means $N(A) = \alpha \Leftrightarrow \Pi(A^c) = \sup_{s \notin A} \pi(s) = 1 - \alpha$
- The least informative π sanctioning $N(A) \geq \alpha$ is :
  - **$\pi(s) = 1$ if $s \in A$ and $1 - \alpha$ if $s \notin A$**
- In other words: $\pi(s) = \max(\mu_A, 1 - \alpha)$

Standard propositional possibilistic logic

- syntax and semantics
- inconsistency handling
- guaranteed possibility-based logic

Possibilistic logic: syntax

- A possibilistic formula is a certainty qualified proposition (p, α), **where p is a classical proposition and** $\alpha \in (0, 1]$ is the minimal certainty of p.
- (p, α) means « p is α-certain » : $N([p]) \geq \alpha$
- A possibilistic knowledge base is a totally pre-ordered logical base = $\mathcal{B}_1 \cup \mathcal{B}_2 \ldots \cup \mathcal{B}_2$
  - **$\mathcal{B}_i = \{(p_{ij}\ \alpha_i), j = 1, \ldots\}$ is the α_i-layer,**
  - **priorities $\alpha_1 > \alpha_2 > \ldots \alpha_m$ lie in some ordinal scale.**

Possibilistic logic: inference

- *Inference in poslog is a straightforward extension of classical inference :* $\mathcal{B}$ |- (p, α) iff $\mathcal{B}_\alpha$ classically implies p : $\mathcal{B}_\alpha$ |- p
- A set of formulas (p_i, α) for any given α is deductively close (wrong for probabilities except if α=1)
- Basic principles
  - *The weight of a chain of inference is the weight of the weakest link*
  - The weight of the conclusion is the *weight of the strongest chain of inference that produces it*

Possibilistic logic: proof method

- *Valid inference patterns*

  Modus ponens: $\{(p, \alpha), (\neg p \vee q, \beta)\} \vdash (q, \min(\alpha, \beta))$

  Resolution: $\{(p \vee q, \alpha), (\neg p \vee r, \beta)\} \vdash (q \vee r, \min(\alpha, \beta))$

  Fusion: $\{(p, \alpha), (p, \beta)\} \vdash (p, \max(\alpha, \beta))$

- if $p \vdash q$ classically, $(p, \alpha) \vdash (q, \alpha)$
- if $\alpha \geq \beta$ then $(p, \alpha) \vdash (p, \beta)$
- *Certainty of a conclusion p*: $\max\{\alpha, \mathcal{B} \vdash (p, \alpha)\}$
- *Degree of contradiction* :

  $$Inc(\mathcal{B}) = \sup\{\alpha : \mathcal{B} \vdash (\bot, \alpha)\}$$

- *Refutation:*

  $$\mathcal{B} \vdash (p, \alpha) \text{ iff } \mathcal{B} \cup (\neg p, 1) \vdash (\bot, \alpha)$$

Possibilistic logic: semantics

A set of sentences $\mathcal{B}$ with priorities models certainty-qualified assertions;

- (p, α) means « x is A is α-certain » : $N(A) \geq \alpha$
- Models of (p, α) form a fuzzy set:
- $\pi_{(p, \alpha)}(s) =$ 1 if s satisfies p ,

  $1 - \alpha$ if s does not satisfy p
- $\mathcal{B}$ is interpreted by the least specific possibility distribution on the set of interpretations obeying the constraints $\{N(A_{ij}) \geq \alpha_i, i = 1, n\}$ where A_{ij} is the set of models of p_{ij}:

$$\pi_{\mathcal{B}} = \min_{ij} \max(\mu_{A_{ij}}, 1 - \alpha_i)$$

SOUNDNESS AND COMPLETENESS

- Semantic inference: $\mathcal{B} \models (p, \alpha)$ means $\pi_{\mathcal{B}} \leq \pi_{(p, \alpha)}$
- $\{(p, \alpha), (q, \alpha)\}$ is semantically equivalent to $\{(p \wedge q, \alpha)\}$: one may put any base in a conjunctive normal form as a set of weighted clauses.
- **Main theorem** : Possibilistic logic is sound and complete w.r.t. this semantics :

$$\mathcal{B} \models (p, \alpha) \text{ iff } \mathcal{B} \vdash (p, \alpha),$$

- An inconsistent $\mathcal{B}$ may yield non-trivial conclusions

Inconsistency-Tolerant inference

- Degree of inconsistency of a possibilistic belief base: $\text{Inc}(\mathcal{B}) = \max\{\alpha, \mathcal{B} \vdash (\bot, \alpha)\}$ $(= 1 - \max_{\omega} \boldsymbol{\pi}_{\mathcal{B}}(\omega))$
  - For all p, $\mathcal{B} \vdash (p, \text{Inc}(\mathcal{B}))$ (trivial part),
- *Inconsistency-Tolerant inference*:

  $\mathcal{B} \vdash_{\text{Pref}} p$ if $\mathcal{B} \vdash (p, \alpha)$ with $\alpha > \text{Inc}(\mathcal{B})$.
- The set of non-trivial consequences of $\mathcal{B}$ are those of the largest set $\{p_{ij} \in \mathcal{B}_1 \cup \mathcal{B}_2 \ldots \mathcal{B}_{\mathbf{i}}\}$ that is not inconsistent ($\text{Inc}(\mathcal{B}) = \alpha_{i+1}$).
- This inference is **non-monotonic** : one may have

$\mathcal{B} \vdash_{\text{Pref}} p$ and $\mathcal{B} \cup (q, 1) \vdash_{\text{Pref}} \neg p$.

Example

- K = {(¬Stu(x) ∨ You(x), a1) (¬You(x) ∨ Ba(x), a2) (¬Stu(x) ∨ ¬ Par(x) ∨ ¬Ba(x), a3) (Stu(Léa), 1)}

  with **a3 > a1**

- Inc(K) = 0 : K |– (Ba(Léa), min(a1,a2))

  K $|-_{Pref}$ (Ba(Léa) (cannot infer ¬Ba(Léa))

- But K ∪ (Par(Léa), 1) is partially inconsistent:

  Inc (K ∪ (Par(Léa), 1)) = min(a1, a2, a3) = min(a1,a2)

- K ∪ {(Par(Léa), 1)} $|-_{Pref}$ ¬Ba(Léa) since

K ∪ {(Par(Léa), 1)} |– (¬Ba(Léa), a3) : nonmon!!!!

and a3 > min(a1,a2).

The Syntactic approach: Conditional assertions

- A |~ B denotes a conditional assertion « generally if A then B » where A and B are propositional sentences.
- Postulates of System P (Kraus et al, 1989)
  - A |~ A (reflexivity)
  - If B ≡ C then C |~ A iff B |~ A (Left logical equivalence)
  - If B |= C then A |~ B implies A |~ C (Right Weakening)
  - If B |~ A and C |~ A then B∪C|~A (OR)
  - A |~ B and A |~ C then A ∩C |~ B (Cautious Monotony)
  - A |~ B and A ∩C |~ B then A |~ C (Cut)
  - If A |~ B and A |~ C then A |~ B∩C (AND)

Belief construction in System P

- *Beliefs of an agent about a situation are inferred from generic knowledge AND observed singular evidence about the case at hand.*
- **Commonsense inference : a two-tiered scheme**
  - Generic knowledge = set of conditional assertions Δ
  - Singular observed facts = proposition A (all you know)
  - Inferred belief = proposition B
  - **Method** : first infer a rule $A \mid\sim B$ (adapted to your singular information A) from Δ, then believe B
  - *Inference of rules in system P is monotonic,*
  - *Inference of beliefs is not :*

may have $\Delta \vdash (A \cap C \mid\sim \neg B)$

Possibilistic logic encoding

- A set of defaults $\Delta = \{Ai \mid\sim Bi\}$ $i = 1,n$
- each $A \mid\sim B$ is associated with the constraint $\Pi(A \cap B) > \Pi(A \cap \neg B)$ iff $\mathbf{N(B \mid A) > 0}$

  with $\mathbf{N(B \mid A) = 1 - \Pi(\neg B \mid A)}$
- Two entailments:

- preferential entailment

For all Π s.t. $\Pi(Ai \cap Bi) > \Pi(Ai \cap \neg Bi)$ $i = 1,n$

we have $\Pi(A \cap B) > \Pi(A \cap \neg B)$

equivalent to $\Delta \vdash A \mid\sim B$

- rational closure

Rational closure

- Compute the largest possibility distribution (it is the least informative)

  corresponding to

  constraints $\Pi(Ai \cap Bi) > \Pi(Ai \cap \neg Bi)$ $i = 1,n$

- $RC(\Delta) = \{A \rightarrow B, \Pi(A \cap B) > \Pi(A \cap \neg B)\}$
- This is rational closure in possibilistic logic

we use pairs $(\neg A \cup B, N(\neg A \cup B))$

Example

- Penguin →Bird, Bird →Flies, Penguin →¬ Flies

  1. $\Pi(P \wedge B) > \Pi(P \wedge \neg B)$ (examples > counterexamples);
  2. $\Pi(B \wedge F) > \Pi(B \wedge \neg F)$;
  3. $\Pi(P \wedge \neg F) > \Pi(P \wedge F)$.

- *Step 1 : Finding Normal cases*

  Exceptional cases are $(P \wedge \neg B) \vee (B \wedge \neg F) \vee (P \wedge F)$

  Normal cases are thus the other models :

  $(\neg P \vee B) \wedge (\neg B \vee F) \wedge (\neg P \vee \neg F)) = \mathbf{\neg P \wedge (\neg B \vee F)}$

  (Non-penguins that, if they are birds, fly).

- Since $(B \wedge F) \wedge \neg P \wedge (\neg B \vee F) = B \wedge F \wedge \neg P \neq \emptyset$, we can give up rule 2.

Example

- *Step 2 : Less normal cases are in* $P \vee (B \vee F)$ and are not exceptions to rules 1 and 3 (i.e., not $(P \wedge \neg F) \vee (P \wedge B)$):

$$\neg((P \wedge \neg F) \vee (P \wedge B)) \wedge (P \vee (B \vee F)) = \mathbf{B \wedge \neg F}$$

(birds that do not fly)

- **Stop** : $B \wedge \neg F$ is consistent with examples $P \wedge B$ et $P \wedge \neg F$.
- *Totally abnormal cases*:

$$\neg[(B \wedge \neg F) \vee (\neg P \wedge (\neg B \vee F)] = \mathbf{P \wedge (\neg B \vee F)}$$

(Penguins that fly, or are not birds)

Back to possibilistic logic

- The well-ordered partition is a possibility distribution:

  $\Pi(\neg \mathbf{P} \wedge (\neg \mathbf{B} \vee \mathbf{F})) > \Pi(\mathbf{B} \wedge \neg \mathbf{F}) > \Pi(\mathbf{P} \wedge (\neg \mathbf{B} \vee \mathbf{F}))$

- For each rule A $\rightarrow$ B define a possibilistic formula

  $(\neg A \vee B, N(\neg A \vee B)) : \mathcal{B}_{\Pi}$

- $N(\neg B \vee F) < N(\neg P \vee B) = N(\neg P \vee \neg F)$
- A $\rightarrow$ B is in RC(Δ) iff $(A, 1) \cup \mathcal{B}_{\Pi} \vdash B$

Reasoning with rational closure

- Any well-ordered partition can be modeled by a set of default under rational closure.
- Non-intuitive conclusions can be repaired by adding the proper default information:
- If RC(Δ) contains a counterintuitive conclusion $A \rightarrow B$, then it is possible to add rules r to Δ so that $RC(\Delta \cup \{r\})$, if not inconsistent, contains $A \rightarrow \neg B$. (Benferhat D&P, Applied Intelligence 1998)

Perceived Causality. An Example

- We were at "...", I was surprised by the person who braked in front of me, not having the option of changing lane and the road being wet, I could not stop completely in time.

- Driver A follows Driver B

- Abnormal facts are privileged when providing causal explanations
- Material implication is insufficient for representing causation
- => Nonmonotonic logic-based approaches for causal ascriptions

Nonmonotonic Consequence Approach

- An agent learns of the sequence $\neg B_t$, A_t , B_{t+k}
- K_t (context):

conjunction of all other facts known by the agent

- $|\sim$ a nonmonotonic consequence relation

(in the sense of System P of Kraus et al., 1990).

Given the sequence ¬B , A , B

- if the agent believes $K \mid\sim \neg B$ and $K \wedge A \mid\sim B$, the agent perceives A to cause B in context K denoted $A \Rightarrow_c B$
- If the agent believes that $K \mid\sim \neg B$ and $K \wedge A \not\mid\sim \neg B$ (rather than $K \wedge A \mid\sim B$), then A is perceived as facilitating B denoted $A \Rightarrow_f B$

- Variables
- Acc (occurrence of an accident)
- Wet (road being wet) ; Sur (A is surprised)
- Brak (driver B brakes in front of driver A)
- ReacL: driver A brakes after B brakes, with a delay
- common core of knowledge is : $|\sim \neg$Acc; $|\sim \neg$Sur; ReacL $\wedge$ Wet $|\sim$ Acc .
- we derive ReacL $\wedge$ Wet $\Rightarrow_c$ Acc.
- cause of the accident s the conjunction of braking late and the road being wet.
- ReacL $|\not\sim \neg$Acc long-delay reacting alone facilitated the accident

- In the definitions of $\Rightarrow_c$ and $\Rightarrow_f$, $|\sim$ is a preferential entailment, and a rational closure entailment,
- respectively Causes and facilitations are abnormal in context:
- If A $\Rightarrow_f$ B or A $\Rightarrow_c$ B then K $|\sim \neg$A.
- Causality is transitive only in particular cases:

If A is the normal way of getting B in context K, i.e.,

K $\wedge$ B $|\sim$ A, and if A $\Rightarrow_c$ B and B $\Rightarrow_c$ C, then A $\Rightarrow_c$ C.

- The distinction between causation and facilitation, as well as the restricted transitivity property, have been validated by behavioral experiments.

Representions of preferences

- different formats

- bipolarity

Possibilistic logic base

- $\mathcal{B} = \{(\mathbf{B_j}, \boldsymbol{\beta_j}), \mathrm{j} = 1, \mathrm{m}\}$

$$\mathbf{N(B_j) \geq \beta_j}$$

$$\mathcal{B} = \{(\mathbf{p_1}, \mathbf{1}), (\mathbf{p_2}, \boldsymbol{\alpha_2}), (\mathbf{p_3}, \boldsymbol{\alpha_3})\}$$

$$\boldsymbol{\pi}_{\mathcal{B}}(\mathrm{d}) = \min(\mu_{P_1}(d), \max(\mu_{P_2}(d), 1-\alpha_2), \max(\mu_{P_3}(d), 1-\alpha_3)).$$

Example

‘near the sea’ **and** ‘affordable price’

‘near’ (the sea)

$\pi_1(u_1) = 1$ if $u_1 \leq \mathbf{5}$;

$= .7$ if $\mathbf{5} < u_1 \leq \mathbf{10}$;

$= .2$ if $\mathbf{10} < u_1 \leq \mathbf{15}$;

$= 0$ if $u_1 > \mathbf{15}$

‘affordable’ (price)

$\pi_2(u_2) = 1$ if $u_2 \leq \mathbf{200}$;

$= .5$ if $\mathbf{200} < u_2 \leq \mathbf{400}$;

$= 0$ if $u_2 > \mathbf{400}$

↳ associated to

$\mathcal{B} = \{(d \leq 15, 1), (d \leq 10, .8), (d \leq 5, .3),$
$(p \leq 400, 1), (p \leq 200, .5)\}$

Example (2)

'near the sea' **or** 'affordable price'

$$\boldsymbol{\pi = \max(\pi_1, \pi_2)}$$

$$\mathcal{B}' = \{(d \le 15 \vee p \le 400, 1), (d \le 10 \vee p \le 400, .8), (d \le 10 \vee p \le 200, .5), (d \le 5 \vee p \le 200, .3)\}$$

- $\mathcal{B}^{tn} = \mathcal{B}^1 \cup \mathcal{B}^2 \cup \{(pi \vee qj, ct(\alpha i, \beta j)) \mid (pi, \alpha i) \in \mathcal{B}^1 \text{ and } (qj, \beta j) \in \mathcal{B}^2\}$,
- $\mathcal{B}^{ct} = \{(pi \vee qj, tn(\alpha i, \beta j)) \mid (pi, \alpha i) \in \mathcal{B}^1 \text{ and } (qj, \beta j) \in \mathcal{B}^2\}$

$$ct(\alpha, \beta) = 1 - tn(1 - \alpha, 1 - \beta)$$

2nd logical reading

- Situations having a guaranteed satisfaction level

Guaranteed possibility

$$\Delta(C) = \min\{\pi(u) \mid u \in C\}$$

$$\Delta(C^{i-1}) \geq \alpha^{i}$$

$\boldsymbol{\pi}$ also equivalent to a set $\{[C^{i-1}, \alpha^{i}], i = 2, n\}$

set of situations $\mathbf{C^{i-1}}$ with guaranteed satisfaction $\boldsymbol{\alpha^{i}}$

$$\forall u \in U, \ \pi_{[C^{i-1}, \alpha^{i}]}(u) = \alpha^{i} \text{ if } u \in C^{i-1}$$

$$\pi_{[C^{i-1}, \alpha^{i}]}(u) = 0 \text{ otherwise}$$

values in $\mathbf{C^{i-1}}$ are possible *at least to* a degree $\boldsymbol{\alpha^{i}}$

Distribution obtained as a

- ***disjunctive*** *combination*

$$\boldsymbol{\pi}(\mathbf{u}) = \mathbf{max}\{\boldsymbol{\pi}_{[C^{i-1}, \alpha^i]}(\mathbf{u}) \mid \mathbf{i} = \mathbf{2}, \mathbf{n}\}$$

$$\mathcal{D} = \{[D_k, \delta_k] \mid k = 1, r\}$$

$$\forall u \in U,\ \boldsymbol{\pi}_{\mathcal{D}}(u) = max\{\boldsymbol{\delta}_{\mathbf{k}} \mid [D_k, \delta_k] \in \mathcal{D} \text{ and } u \in D_k\}$$
$$\text{if } u \in D_1 \cup \ldots \cup D_r$$

$$\boldsymbol{\pi}_{\mathcal{D}}(u) = 0 \text{ otherwise}$$

Example (continued)

‘near the sea’ **and** ‘affordable price’

$$\boldsymbol{\pi = \min(\pi_1, \pi_2)}$$

$\mathcal{D} = \{[d \leq 5 \wedge p \leq 200\ , 1], [5 < d \leq 10 \wedge p \leq 200, .7], [d \leq 10 \wedge 200 < p \leq 400, .5], [10 < d \leq 15 \wedge p \leq 400, .2]\}$

‘near the sea’ **or** ‘affordable price’

$$\boldsymbol{\pi = \max(\pi_1, \pi_2)}$$

$\mathcal{D}' = \{[d \leq 5, 1], [d \leq 10, .7], [d \leq 15\ , .2], [p \leq 400, .5], [p \leq 200, 1]\}$

Conditional preferences

"I prefer to take a tea (**t**).
If there is no tea then I will take a coffee (**c**)"

$\Pi(t) > \Pi(\neg t)$

$\Pi(\neg t \wedge c) > \Pi(\neg t \wedge \neg c)$

There exists a *unique possibility distribution* which is *minimally specific* and satisfies a given set of consistent constraints (such as the above ones)

$\pi(ct) = 1$; $\pi(\neg ct) = 1$; $\pi(c\neg t) = \alpha$;
$\pi(\neg c\neg t) = \beta$ with $\alpha > \beta$

associated to **N**− anf **Δ**−type possibilistic bases :

↳ $\mathcal{B} = \{(c \vee t, 1-\beta), (t, 1-\alpha)\}$

↳ $\mathcal{D} = \{[t, 1], [c\wedge\neg t, \alpha], [\neg c\wedge\neg t, \beta]\}$

one can go from a representation format to another

Graphical representation

Graphical encoding by a *possibilistic Bayesian network*

- $\pi(t) = 1 \qquad \pi(\neg t) = 1$
- $\pi(c \mid \neg t) = \lambda \quad \pi(\neg c \mid \neg t) = 0$
  $\pi(c \mid t) = 1 \qquad \pi(\neg c \mid t) = 1$

$$\boldsymbol{\pi(x, y) = \min(\pi(y \mid x), \pi(x))}$$

conditional *non-interactivity*

translation procedures without loss of information

CP-nets Motivating Example

(P1): he prefers black vest to white vest $\{V_b, V_w\}$
(P2): he prefers black pants to white pants $\{P_b, P_w\}$
(P3): when vest and pants have the *same* color,
he prefers red shirt to white shirt
otherwise he prefers white shirt $\{S_r, S_w\}$
(P4): when the shirt is red then he prefers red shoes
otherwise he prefers white shoes $\{C_r, C_w\}$

$$\Omega =$$

$$\{V_bP_bS_rC_r, V_bP_bS_wC_r, V_bP_wS_rC_r, V_bP_wS_wC_r,$$
$$V_wP_bS_rC_r, V_wP_bS_wC_r, V_wP_wS_rC_r, V_wP_wS_wC_r,$$
$$V_bP_bS_rC_w, V_bP_bS_wC_w, V_bP_wS_rC_w, V_bP_wS_wC_w,$$
$$V_wP_bS_rC_w, V_wP_bS_wC_w, V_wP_wS_rC_w, V_wP_wS_wC_w\}$$

P1: $\{(Vb, 1 - \alpha)\}$

P2: $\{(Pb, 1 - \beta)\}$

P3: $\{(\neg Vb \vee \neg Pb \vee Sr, 1 - \gamma),$

$(\neg Vw \vee \neg Pw \vee Sr, 1 - \eta),$

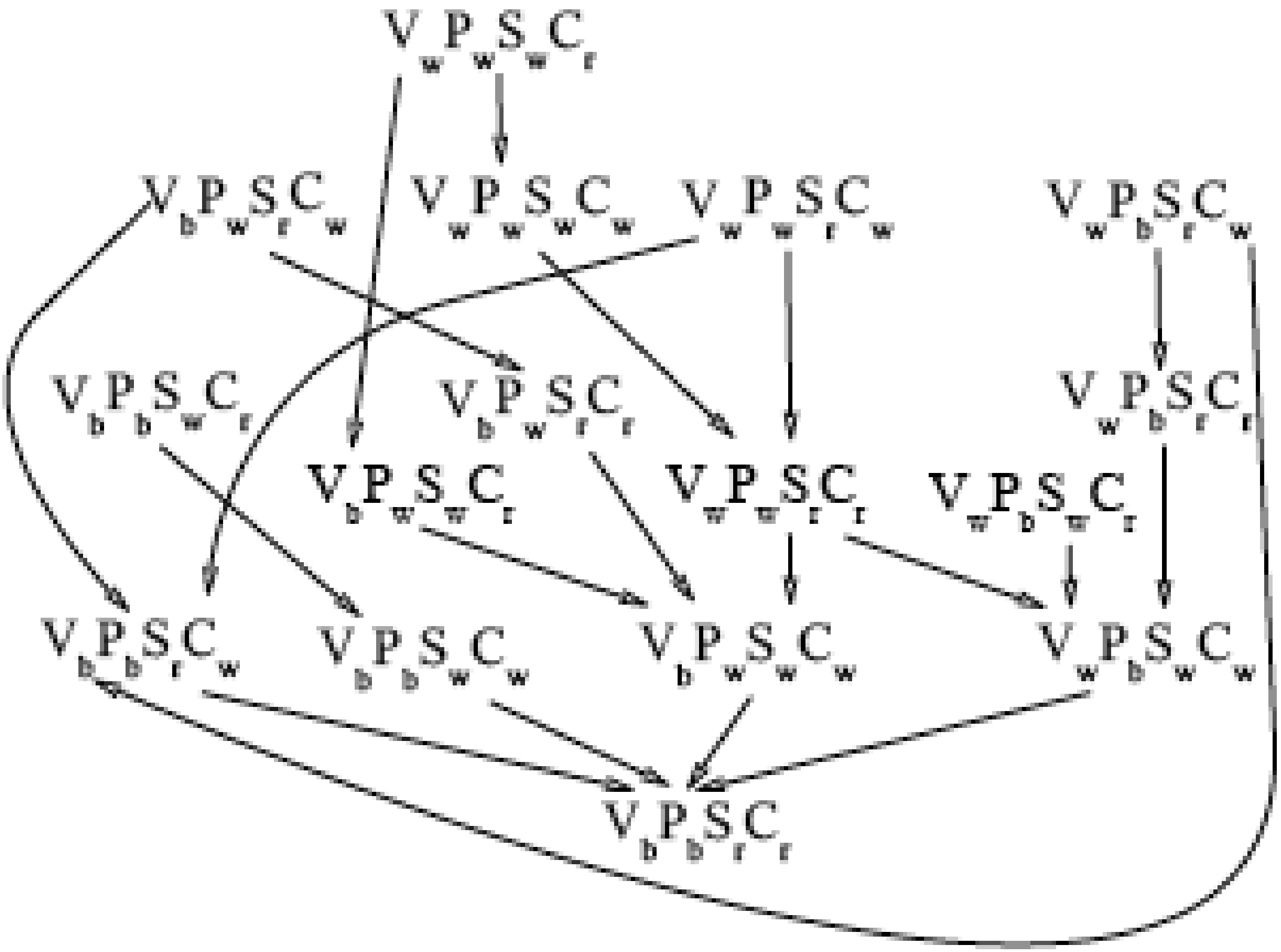
$(\neg Vw \vee \neg Pb \vee Sw, 1 - \delta),$

$(\neg Vb \vee \neg Pw \vee Sw, 1 - \varepsilon)\}$

P4: $\{(\neg Sr \vee Cr, 1 - \theta), (\neg Sw \vee Cw, 1 - \varrho)\}$

- assumed to belong to a *linearly ordered scale* $1 > 1-\alpha \quad \alpha>0$
- $1 - \alpha$, $1 - \beta$, $1 - \gamma$, $1 - \eta$, $1 - \delta$, $1 - \varepsilon$, $1 - \theta$, $1 - \varrho$ are **unknown**
- no particular ordering is assumed between them

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	$(viii)$	satisfaction levels
$V_bP_bS_rC_r$	1	1	1	1	1	1	1	1	$(1,1,1,1,1,1,1,1)$
$V_bP_bS_wC_r$	1	1	γ	1	1	1	1	ρ	$(1,1,\gamma,1,1,1,1,\rho)$
$V_bP_wS_rC_r$	1	β	1	1	1	ε	1	1	$(1,\beta,1,1,1,\varepsilon,1,1)$
$V_bP_wS_wC_r$	1	β	1	1	1	1	1	ρ	$(1,\beta,1,1,1,1,1,\rho)$
$V_wP_bS_rC_r$	α	1	1	1	δ	1	1	1	$(\alpha,1,1,1,\delta,1,1,1)$
$V_wP_bS_wC_r$	α	1	1	1	1	1	1	ρ	$(\alpha,1,1,1,1,1,1,\rho)$
$V_wP_wS_rC_r$	α	β	1	1	1	1	1	1	$(\alpha,\beta,1,1,1,1,1,1)$
$V_wP_wS_wC_r$	α	β	1	η	1	1	1	ρ	$(\alpha,\beta,1,\eta,1,1,1,\rho)$
$V_bP_bS_rC_w$	1	1	1	1	1	1	θ	1	$(1,1,1,1,1,1,\theta,1)$
$V_bP_bS_wC_w$	1	1	γ	1	1	1	1	1	$(1,1,\gamma,1,1,1,1,1)$
$V_bP_wS_rC_w$	1	β	1	1	1	ε	θ	1	$(1,\beta,1,1,1,\varepsilon,\theta,1)$
$V_bP_wS_wC_w$	1	β	1	1	1	1	1	1	$(1,\beta,1,1,1,1,1,1)$
$V_wP_bS_rC_w$	α	1	1	1	δ	1	θ	1	$(\alpha,1,1,1,\delta,1,\theta,1)$
$V_wP_bS_wC_w$	α	1	1	1	1	1	1	1	$(\alpha,1,1,1,1,1,1,1)$
$V_wP_wS_rC_w$	α	β	1	1	1	1	θ	1	$(\alpha,\beta,1,1,1,1,\theta,1)$
$V_wP_wS_wC_w$	α	β	1	η	1	1	1	1	$(\alpha,\beta,1,\eta,1,1,1,1)$


$V_wP_wS_wC_r$
$V_bP_wS_rC_w$
$V_wP_wS_wC_w$
$V_wP_wS_rC_w$
$V_wP_bS_rC_w$
$V_bP_bS_wC_r$
$V_bP_wS_rC_r$
$V_wP_bS_rC_r$
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$V_wP_wS_rC_r$
$V_wP_bS_wC_r$
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$V_bP_bS_wC_w$
$V_bP_wS_wC_w$
$V_wP_bS_wC_w$
$V_bP_bS_rC_r$

Bipolar preferences

Positive information refers to what is *desired*

Negative information refers to what is *rejected*

Pair of possibility distributions (π_*, π^*)

π_* fuzzy set of values *guaranteed to be satisfactory*

π^* evaluates what is *non-impossible*

$1 - \pi^*(u)$ evaluates the extent to which **u** is **impossible**

coherence condition

for the pair (π_*, π^*) :

$$\forall u, \pi_*(u) \leq \pi^*(u)$$

$$\mathcal{B}^* = \{(p_i, \alpha_i), i = 1, n\}$$

and

$$\mathcal{D}_* = \{[q_j, \gamma_j], j = 1, m\}$$

Application to flexible queries

distinction is made between

constraints, whose violation has a *negative* effect,

and

wishes to satisfy if possible,
whose satisfaction has a *positive* effect
(non satisfaction has no impact on the evaluation)

symbolic optimization problem

Reasoning with bipolar knowledge

N - Resolution:

$$\{(p \vee q, \alpha), (\neg p \vee r, \beta)\} \vdash (q \vee r, \min(\alpha, \beta))$$

Δ - Resolution:

$$[p \wedge q, \alpha], [\neg p \wedge r, \beta] \vdash [q \wedge r, \min(\alpha, \beta)]$$

Deductive bipolar reasoning

$$\text{rules : } \textit{if } X \textit{ is } A_i \textit{ then } Y \textit{ is } B_i$$

express that

- Situations where *X is* A_i and *Y is not-*B_i are *impossible*

not A_i ***or*** B_i

conjunctive combination of rules :

$B' = A' \circ \bigcap_i (A_i \rightarrow B_i)$ $\qquad B' = B_i \quad \text{if } A' = A_i$

- Situations where *X is* A_i and *Y is* B_i are guaranted possible

A_i ***and*** B_i

disjunctive combination of rules : $\bigcup_i (A_i \times B_i)$

$B' = \{y \text{ s.t. } \forall x \in A' \text{ and } (x,y) \in \bigcup_i (A_i \times B_i)\}$

$B' = B_i \text{ if } A' = A_i$

- Example:

- **R1**: *if an employee is in **category 1** then his salary is*

  *necessarily in **[1000, 2000]***

  *typically in **[1500, 1800]***

- **R2**: : *if an employee is in **category 2** then his salary is*

  *necessarily in **[1500, 2500]***

  *typically in **[1700, 2000]**.*

* $B' = A' \circ \bigcap_i (A_i \rightarrow B_i)$ **A' = {cat.1, cat.2}**

$A_1 = \{cat.1\}, B_1 = [1000, 2000]$

$A_2 = \{cat.2\}, B_2 = [1500, 2500]$

$\Rightarrow \quad \mathbf{B' = B_1 \cup B_2} = [1000, 2500]$

* $B' = \{y \text{ s.t. } \forall x \in A' \text{ and } (x,y) \in \bigcup_i (A_i \times B_i)\}$,

$B_1 = [1500, 1800], B_2 = [1700, 2000]$,

$\Rightarrow \mathbf{B' = B_1 \cap B_2} = [1700, 1800]$ *guaranted possible*

Reasoning with possibilistic lower bounds in possibilistic logic

formula $\langle p, a \rangle$

encoding the contrainte $\Pi(p) \geq a$

Mixed resolution rule:

$$(\neg p \vee q, a); \langle p \vee r, b \rangle \mid\!\!-\!- \langle q \vee r, b \rangle \quad \text{if } b > 1 - a$$

Reasoning about ignorance

Multiple-agent extension of possibilistic logic

- **Multiple-agent extension of possibilistic logic**

- **Modeling epistemic states in generalized possibilistic logic**

Generalized possibilistic logic and **ASP**

- Generalized possibilistic logic can capture
  logic programing
- with **negation as a failure**,
- "**q is certain provided that p is certain
  and that one has no proof of r**"
- i.e. $\text{if } N(p) \geq a \text{ and } \Pi(\neg r) \geq b \text{ then } N(q) \geq a$
- Which corresponds to formula

$$\neg(p, a) \vee \neg\langle\neg r, b\rangle \vee (q, a)$$

Nested formula

(p, α) is true ou false!

possibilistic knowledge base **K**

- either $N_K(p) \geq \alpha$

  (p, α) holds as (certainly) true
- either $N_K(p) < \alpha$ and (p, α) is false

((p, α), β) ?

possibility distribution over possibility distributions (Zadeh 1978)

- possible at level **1** that the correct representation of information is

$$\pi = \pi_{\{(p,\ \alpha)\}} = \max(\mu_{[p]}, 1 - \alpha)$$

- possible at level **1 − β** that correct representation of information is $\pi = 1$ everywhere (complete *ignorance)*

$$\pi = \max(\min(\pi_{\{(p,\alpha)\}}, 1), \min(1, 1-\beta)) = \max(\mu_{[p]}, 1 - \min(\alpha, \beta))$$

((p, α), β) equivalent to **(p, min(α, β))** (discounting)

counterpart of identity $\Box\Box p \equiv \Box p$ S5

Other applications

- Information fusion, preferences fusion
- Reasoning under inconsistency
- Expressing qualitative independence
- Qualitative decision under uncertainty
  - pessimistic criterion, optimistic criterion
- Logical reprentation of a Sugeno integral

- *Formal concept analysis*

Conclusion

- The setting of *possibilistic logic* is suitable for handling a large number of issues in knowledge representation in AI
- close to *classical logic*, rich *modal* language

- Besides, in *quantitative* possibility theory
  - a *possibility distribution* represents a ***family** of probability distributions*
  - imprecise regression
  - possibility theory *complementary* to probability theory