Structure of oppositions, formal concept analysis and back to possibility theory

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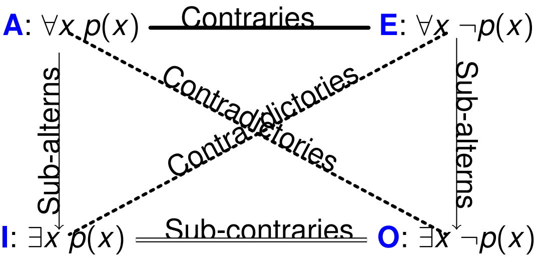
Structures of opposition as a unifying framework

- Structures of opposition
  - Square
  - Hexagon
  - Cube
- Formal Concept Analysis
- Rough Set Theory

The square of oppositions



Aristotle

AffIrmo / NegO

Another instance:

A: $\Box p$ **E**: $\Box\neg p$ **I**: $\Diamond p$ **O**: $\Diamond\neg p$

where $\Diamond p =_{def} \neg\Box\neg p$

(with $p \neq \bot, \top$)

The hexagon of opposition

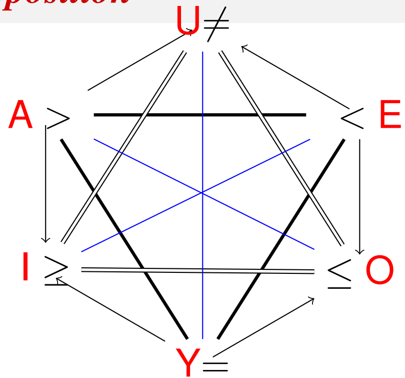


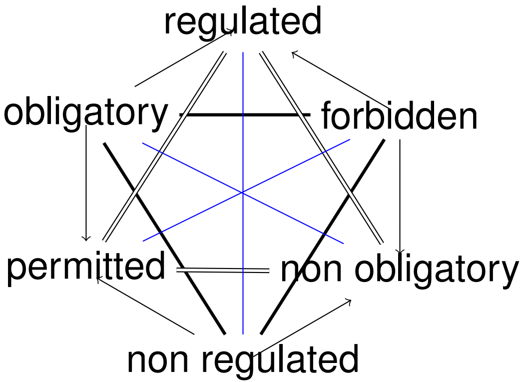
Figura: Hexagon induced by a complete preorder (Robert Blanché, 1953)

$$U = A \vee E \quad Y = I \wedge O$$

Three squares *(A. Sesmat, 1951)*

Renewal of interest with the work of Jean-Yves Béziau

Another hexagon



Why hexagons ?

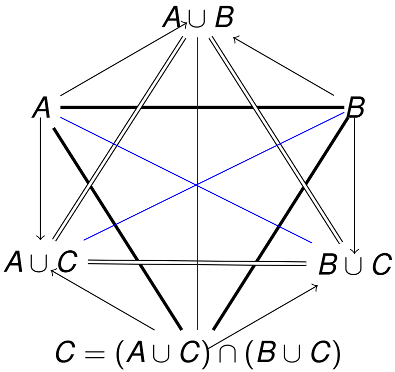
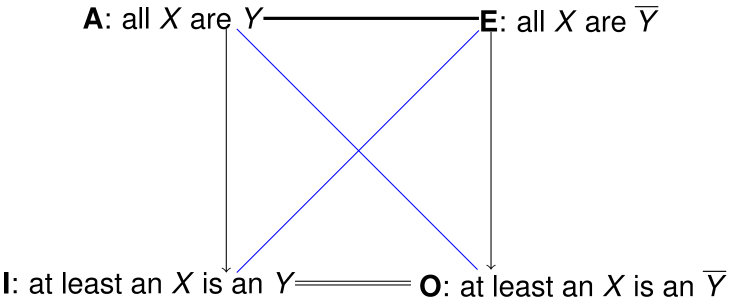
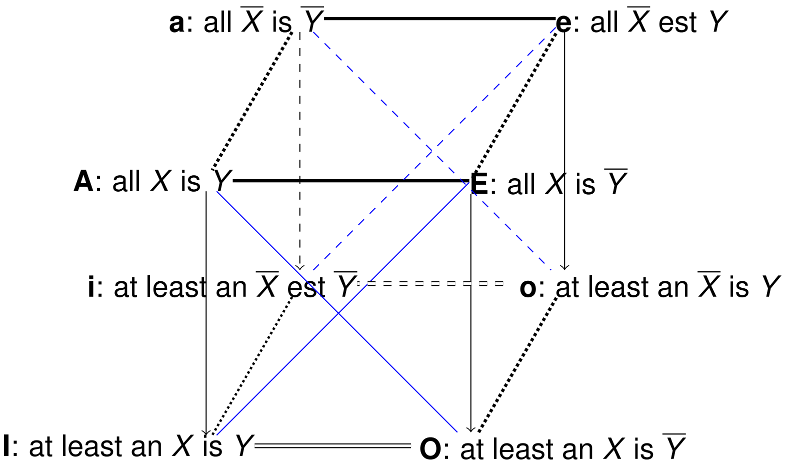


Figura: Hexagon induced by a **tri-partition** (A, B, C)

From square to cube



Cube of opposition



Piaget's group

Klein's group of logical transformations with 4 elements

logical formula $\phi = f(p, q, r, ...)$

- identity $I(\phi) = \phi$
- negation $N(\phi) = \neg\phi$
- reciprocation $R(\phi) = f(\neg p, \neg q, \neg r, ...)$
- correlation $C(\phi) = \neg f(\neg p, \neg q, \neg r, ...)$
- $N = RC$, $R = NC$, $C = NR$, et $I = NRC$

at work in the two diagonal rectangles AaOo and EeIi

A Relation R and a Subset S

binary relation $R \neq \emptyset$ on $X \times Y$ (one may have $Y = X$)

$$xR = \{y \in Y | (x, y) \in R\}$$

normalization assumption $\forall x \ xR \neq \emptyset$

we write xRy for $(x, y) \in R$, and $\neg(xRy)$ for $(x, y) \notin R$

subset $S \subseteq Y$

It gives birth to the two subsets

$$R(S) = \{x \in X | \exists s \in S, xRs\} = \{x \in X | \ S \cap xR \neq \emptyset\}$$

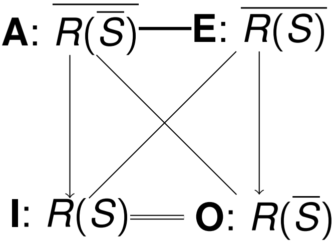
$$R(\overline{S}) = \{x \in X \mid \exists s \in \overline{S}, xRs\}$$

and their complements

$$\overline{R(S)} = \{x \in X \mid \forall s \in S, \neg(xRs)\}$$

$$\overline{R(\overline{S})} = \{x \in X | \forall s \in \overline{S}, \neg(xRs)\} = \{x \in X | \ xR \subseteq S\}$$

A square of opposition, as the square of modalities



- $\overline{R(\overline{S})}$ and $R(\overline{S})$ are complements, as $\overline{R(S)}$ and $R(S)$ assuming the *X-normalization condition* $\forall x, \; xR \neq \emptyset$:
- $\overline{R(\overline{S})} \subseteq R(S)$, and $\overline{R(S)} \subseteq R(\overline{S})$
- $\overline{R(\overline{S})} \cap \overline{R(S)} = \emptyset$; one may have $\overline{R(\overline{S})} \cup \overline{R(S)} \neq Y$
- $R(S) \cup R(\overline{S}) = X$; one may have $R(S) \cap R(\overline{S}) \neq \emptyset$

The complementary relation $\overline{R}$

$x\overline{R}y$ iff $\neg(xRy)$ $\overline{R} \neq \emptyset$ (i.e., $R \neq X \times Y$)
assume the X-normalization of $\overline{R}$, i.e. $\forall x, \exists y\ \neg(xRy)$

We get 4 other subsets of X from $\overline{R}$

$$\overline{R}(\overline{S}) = \{x \in X | \exists s \in \overline{S}, \neg(xRs)\} = \{x \in X\ | S \cup xR \neq X\}$$
$$\overline{R}(S) = \{x \in X \mid \exists s \in S, \neg(xRs)\}$$

and their complements

$$\overline{\overline{R}(\overline{S})} = \{x \in X \mid \forall s \notin S, xRs\}$$
$$\overline{\overline{R}(S)} = \{x \in X | \forall s \in S, xRs\} = \{x \in X \mid S \subseteq xR\}$$

The 8 subsets can be organized into a cube of oppositions

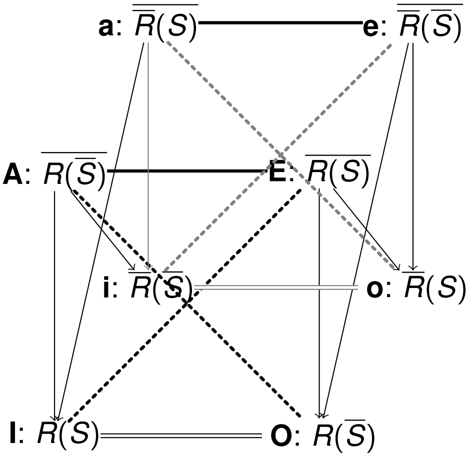


Figura: Cube of oppositions induced by a relation R and a subset S

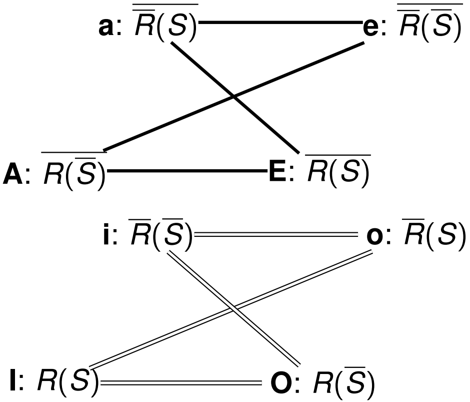


Figura: Top and bottom facets of the cube of oppositions

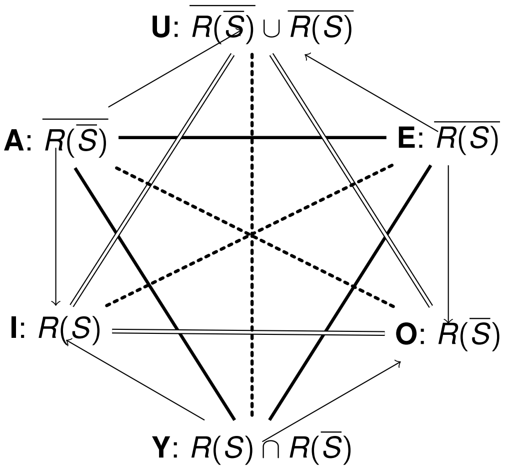


Figura: Hexagon associated with the front facet of the cube

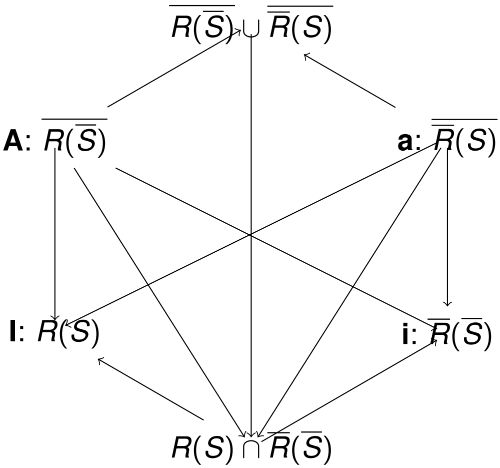


Figura: Hexagon induced by the left-hand side square

(Generalized) formal concept analysis

- relation $R \subseteq X \times Y = Obj \times Prop$ *formal context*
  $R(x)$ the set of properties possessed by an object x, et
  $R^{-1}(y)$ the set of objects having property y.
- $R^{\Pi}(S) = \{x \in Obj | xR \cap S \neq \emptyset\} = \cup_{y \in S} Ry$
  set of objects having at least a property in S
- $R^{N}(S) = \{x \in Obj | xR \subseteq S\} = \cap_{y \notin S} \overline{Ry}$
  set of objects having none property outside S
- $R^{\Delta}(S) = \{x \in Obj | xR \supseteq S\} = \cap_{y \in S} Ry$
  set of objects having all the properties in S
- $R^{\nabla}(S) = \{x \in Obj | xR \cup S \neq Prop\} = \cup_{y \notin S} \overline{Ry}$
  set of objects to which at least a property outside S is missing

Example : $R^{\sqcap}(\{5, 6\}) = \{a, b, c, d, f\}$

Objects satisfying at least 5 or 6

properties

	1	2	3	4	**5**	**6**	7	8
a					$\otimes$	$\otimes$	×	×
b					$\otimes$	$\otimes$		
c						$\otimes$	×	×
d					$\otimes$	$\otimes$	×	×
e							×	
f					$\otimes$	$\otimes$		×
g	×	×	×	×				
h		×	×	×				
i				×				

Example : $R^N(\{5, 6\}) = \{b\}$

Objects satisfying 5 and 6 and no other property

properties

	1	2	3	4	**5**	**6**	7	8
a					×	×	×	×
b					⊗	⊗		
c						×	×	×
d					×	×	×	×
e							×	
f					×	×		×
g	×	×	×	×				
h		×	×	×				
i				×				

Example : $R^{\Delta}(\{5,6\}) = \{a,b,d,f\}$

Objects satisfying at least both 5 and 6

properties

	1	2	3	4	**5**	**6**	7	8
a					$\otimes$	$\otimes$	$\times$	$\times$
b					$\otimes$	$\otimes$		
c						$\times$	$\times$	$\times$
d					$\otimes$	$\otimes$	$\times$	$\times$
e							$\times$	
f					$\otimes$	$\otimes$		$\times$
g	$\times$	$\times$	$\times$	$\times$				
h		$\times$	$\times$	$\times$				
i				$\times$				

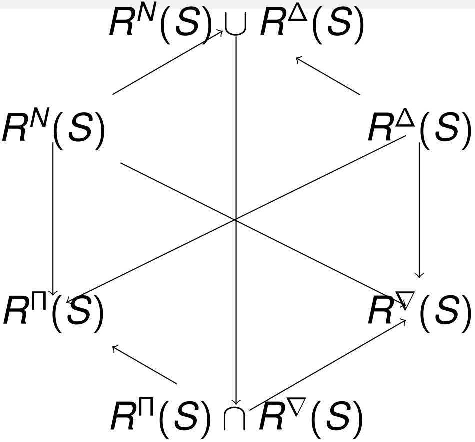
Example : $R^{\nabla}(\{5,6\}) = \{a,b,c,d,e,f,g,h,i\}$

Objects missing at least one property other than 5 and 6

properties

	1	2	3	4	**5**	**6**	7	8
a					$\times$	$\times$	$\otimes$	$\otimes$
b					$\times$	$\times$		
c						$\times$	$\otimes$	$\otimes$
d					$\times$	$\times$	$\otimes$	$\otimes$
e							$\otimes$	
f					$\times$	$\times$		$\otimes$
g	$\otimes$	$\otimes$	$\otimes$	$\otimes$				
h		$\otimes$	$\otimes$	$\otimes$				
i				$\otimes$				

The hexagon of formal concept analysis



$$R^N(S) \cup R^\Delta(S) \subseteq R^\Pi(S) \cap R^\nabla(S)$$

A well-known Galois Connection: formal concepts

Def. : A *formal concept* is a pair (T, S) of extent and intent
such that $R^{\Delta}(T) = S$ and $R^{-1\Delta}(S) = T$
(Ganter and Wille and Barbut and Montjardet)

- This is equivalent to finding a pair of largest sets (T, S) such that $T \times S \subseteq R$.
- All objects in T have all properties in S, all properties in S are satisfied by all objects in T.
- $R^{\nabla}(T) = S$ and $R^{-1\nabla}(S) = T$ if and only if $(\overline{T}, \overline{S})$ is a formal concept.

Example

Formal concepts: ({g, h}, {2, 3, 4});({a, b, d, f}, {5, 6});({a, c, d}, {6, 7, 8}).

properties

	1	2	3	4	5	6	7	8
a					$\oplus$	$\oplus\odot$	$\odot$	$\odot$
b					$\oplus$	$\oplus$		
c						$\odot$	$\odot$	$\odot$
d					$\oplus$	$\oplus\odot$	$\odot$	$\odot$
e							$\times$	
f					$\oplus$	$\oplus$		$\times$
g	$\times$	$\otimes$	$\otimes$	$\otimes$				
h		$\otimes$	$\otimes$	$\otimes$				
i				$\times$				

Another correspondence: conjugated pairs

- $R^{\Pi}(T) = S$ and $R^{-1\Pi}(S) = T$;
- Conjugated pairs are *independent subcontexts*
- $R \subseteq (T \times S) \cup (\overline{T} \times \overline{S})$.

Example

Formal subcontexts : ({g, h, i}, {1,2, 3, 4});({a, b, c, d,e, f}, {5, 6, 7, 8})

properties

	1	2	3	4	5	6	7	8
a					⊙	⊙	⊙	⊙
b					⊙	⊙		
c						⊙	⊙	⊙
d					⊙	⊙	⊙	⊙
e							⊙	
f					⊙	⊙		⊙
g	×	×	×	×				
h		×	×	×				
i				×				

Possibility theory

- i) *(weak) possibility* *measure*
  $\Pi(A) = \max_{u \in A} \pi(u)$
  $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$
- ii) dual *(strong) necessity* measure
  $N(A) = \min_{u \notin A} 1 - \pi(u) = 1 - \Pi(\overline{A})$
  $N(A \cap B) = \min(N(A), N(B))$
- iii) (strong) possibility measure
  $\Delta(A) = \min_{u \in A} \pi(u)$
  $\Delta(A \cup B) = \min(\Delta(A), \Delta(B))$
- iv) dual *(weak)* necessity measure
  $\nabla(A) = \max_{u \notin A} 1 - \pi(u) = 1 - \Delta(\overline{A})$
  $\nabla(A \cap B) = \max(\nabla(A), \nabla(B))$

Boolean Posssibility theory

Let $E \subset X$ be a *non-empty* proper subset of possible situations in a larger set of states X.
Another subset (event) A is

- **potentially possible** if $A \cap E \neq \emptyset : \Pi(A) = 1(0 \text{ otherwise})$.
  if $x \in E$ then it is possibly in A.
- **actually possible** if $A \subseteq E : \Delta(A) = 1(0 \text{ otherwise})$.
  it is enough that $x \in A$ to be sure x is possible.
- **actually necessary** if $E \subseteq A : N(A) = 1(0 \text{ otherwise})$.
  if $x \in E$ then it is surely in A.
- **potentially necessary** if $A \cup E \neq S : \nabla(A) = 1(0 \text{ otherwise})$.
  if $x \notin E$ then it is possibly not in A

One has $\max(N(A), \Delta(A)) \leq \min(\Pi(A), \nabla(A))$:
only 7 Boolean 4-tuples $(N(A), \Delta(A), \Pi(A), \nabla(A))$ out of 16.

Encoding the relative position of sets

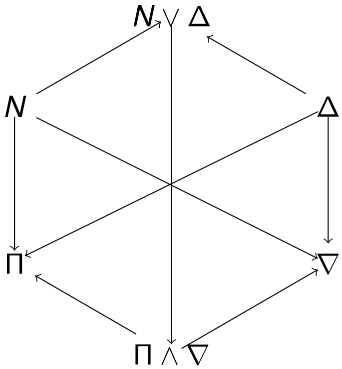
- There are 7 possible relative positions of $E \neq \emptyset$, S and A :

Position	Π	Δ	N	∇
$A = \overline{E}$	0	0	0	0
$A \subset \overline{E}$	0	0	0	1
$\overline{E} \subset A$	1	0	0	0
Pure overlap	1	0	0	1
$E \subset A$	1	0	1	1
$A \subset E$	1	1	0	1
$A = E$	1	1	1	1

Tabella: Relative position of sets

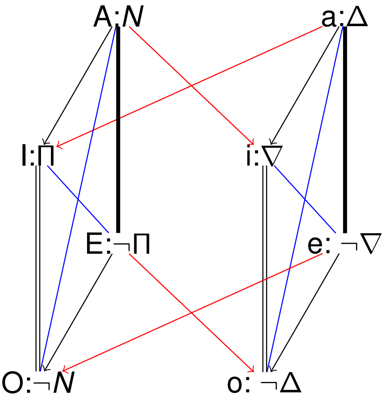
Possibilistic hexagon

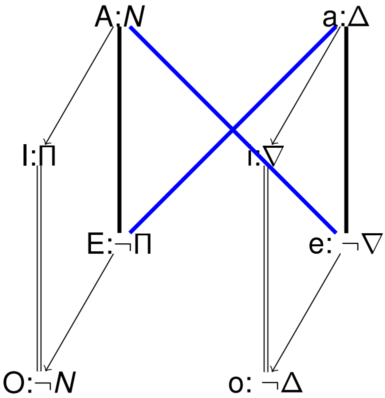
$$R^N(Y) \cup R^\Delta(Y) \subseteq R^\Pi(Y) \cap R^\nabla(Y)$$



all lines express implications

Cube of possibility theory





Conclusion - 1

- It can be shown that both RST (rough set theory) and FCA have the same type of underlying structure: the cube of oppositions
- We have pointed out how having in mind this structure may lead to substantially enlarge the theoretical settings of both RST and FCA
- This helped us to provide an organized view of the related literature and to suggest new directions worth investigating
- Such a structured view, which also includes possibility theory (and modal logic), may contribute to the foundations of a basic framework for information processing

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Conclusion - 2

- square of oppositions closely related to the study of syllogisms
- a cube of oppositions
- 3 theories developed for 30 years
  - for analyzing relations between objects and properties
  - for handling indiscernible objects, and
  - for modeling epistemic uncertainty
  have their roots in the square of oppositions
- **fuzzy relations**
  in formal concept analysis and rough sets
- structures of opposition useful in argumentation