Multiple agent possibilistic logic
Generalized possibilistic logic

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Introduction

- reasoning about pieces of (uncertain) information

  held by subgroups of agents

  (p, A) "all agents in A are certain that p is true"

- *not so much* to try to take the best of the information provided by sets of agents viewed as sources as in fusion

  rather to understand what claims a group of agents supports *with what other groups they are in conflict*, about what

- to distinguish the individual inconsistency of agents from the global inconsistency of a group of agents

Contents

- multiple-agent logic
- *multiple-agent* possibilistic logic
- generalized possibilistic logic

Multiple-agent logic - Syntax

- pairs (p_i, A_i) p_i proposition $A_i \neq \emptyset$ subset

  of agents $A_i \subseteq ALL$

- multiple-agent logic base = conjunction of such pairs

- $(\neg p \vee q, A), (p \vee r, B) \vdash (q \vee r, A \cap B))$

- inconsistency of K: $inc(K) = \cup\{A | K \vdash (\bot, A)\}$

- $inc(K)$ subset of the agents individually inconsistent

- one may have $inc(K) = \emptyset$ even if K^* is inconsistent

  $K^* = \{p_i | (p_i, A_i) \in K\}$

- Example $K = \{(p, B), (\neg p, \overline{B})\}$

Multiple-agent logic - Semantics

- (p_i, A_i) $\mathbf{N}(p_i) \supseteq A_i$

  set necessity $\mathbf{N}(p \wedge q) = \mathbf{N}(p) \cap \mathbf{N}(q)$

- $\mathbf{N}(p) = \overline{\mathbf{\Pi}(\neg p)}$ and $\mathbf{\Pi}(p) = \bigcup_{\omega:\ \omega \models p} \pi_K(\omega)$

- ***set-valued*** possibility distribution $\pi_K(\omega) =$

  $\pi_{\{(p_i, A_i) | i=1,m\}}(\omega) = \bigcap_{i=1,m}([p_i](\omega) \cup \overline{A_i}))$

  $[p_i](\omega) = ALL$ if $\omega \models p_i$; $[p_i](\omega) = \emptyset$ otherwise

- $K \models (p, A)$ iff $\forall \omega, \pi_K(\omega) \subseteq \pi_{\{(p,A)\}}(\omega)$

- $inc(K) = \cap_\omega \overline{\pi_K(\omega)}$ $inc(K) = \emptyset$ weaker than

$\exists \omega, \pi_K(\omega) = ALL$: the agents are collectively consistent

Standard possibilistic logic - Syntax

- pairs (p_i, α_i) p_i proposition α_i certainty level
- standard possibilistic base = conjunction of such pairs
- $(\neg p \vee q, \alpha), (p \vee r, \beta) \vdash (q \vee r, \min(\alpha, \beta))$
- inconsistency level of a base K:

  $inc(K) = \max\{\alpha | K \vdash (\bot, \alpha)\}$
- $inc(K) = 0$ iff K^* is consistent $K^* = \{p_i | (p_i, \alpha_i) \in K\}$
- $K \vdash (p, \alpha)$ iff $K^*_\alpha \vdash p$ and $\alpha > inc(K)$

  $K^*_\alpha = \{(p_i, \alpha_i) \in K, \alpha_i \geq \alpha\}$

Standard possibilistic logic - Semantics

- (p_i, α_i) $\quad N(p_i) \geq \alpha_i$

  necessity $N(p \wedge q) = \min(N(p), N(q))$

- $N(p) = 1 - \Pi(\neg p)$ and $\Pi(p) = \max_{\omega:\ \omega \models p} \pi_K(\omega)$

- possibility distribution

  $\pi_K(\omega) = \pi_{\{(p_i,\alpha_i) | i=1,m\}}(\omega)$

  $= \min_{i=1,m} \max([p_i](\omega), 1 - \alpha_i)$

  $[p_i](\omega) = 1$ if $\omega \models p_i$; $[p_i](\omega) = 0$ otherwise

- $K \models (p, \alpha)$ iff $\forall \omega, \pi_K(\omega) \leq \pi_{\{(p,\alpha)\}}(\omega)$

- $inc(K) = 1 - \max_{\omega}\ \pi_K(\omega)$

Multiple-agent possibilistic logic. Syntax

- pairs $(p_i, \alpha_i / A_i)$ p_i prop., α_i certainty level, A_i subs. agents
- Multiple-agent possibilistic logic base: conjunction of such pairs
- $(\neg p \vee q, \alpha / A), (p \vee r, \beta / B) \vdash (q \vee r, \min(\alpha, \beta) / A \cap B)$
- inconsistency level of a base K:
  $inc(K) = \cup\{\alpha / A \mid K \vdash (\bot, \alpha / A)\}$
- $inc(K)$ fuzzy subset of agents individually inconsistent

Multiple-agent possibilistic logic - Semantics

- $(p_i, \alpha_i/A_i)$ $\quad\quad$ $\mathbf{N}(p_i) \supseteq \alpha_i/A_i$

  $\alpha_i/A_i(a) = \alpha_i$ if $a_i \in A_i$ et $\alpha_i/A_i(a) = 0$ si $a_i \notin A_i$

  more generally $(p_i, \bigcup_j \alpha_{i,j}/A_{ij})$

  ***fuzzy* set-valued** necessity $\mathbf{N}(p \wedge q) = \mathbf{N}(p) \cap \mathbf{N}(q)$

- $\mathbf{N}(p) = \overline{\mathbf{\Pi}(\neg p)}$ and $\mathbf{\Pi}(p) = \cup_{\omega:\ \omega \models p} \pi_K(\omega)$

- $inc(K)$ describes to what extent

  different subsets of agents are inconsistent

  to different degrees

Conclusion

- Multiple agent possibilistic logic

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- extensions

  at most the agents in A believe p

  at least one agent in A believes p

  generalized possibilistic logic

Generalized possibilistic logic

Possibilistic logic : epistemic semantics

Alternatively, we can consider satisfiability of a possibilistic formula by a possibility distribution on Ω

- For an epistemic state π : $\pi \models (p, \alpha)$ if and only if $N(p) \geq \alpha$ (this is known as "forcing").
- The set of (meta-)models of (p, α) is denoted by $\mathbf{Pi}((p, \alpha)) = \{\pi : \pi \models (p, \alpha)\}$.
- $\pi \models B$ iff $\pi \models (p, \alpha), \forall (p, \alpha) \in B$: $\mathbf{Pi}(B) = \bigcap_{(p,\alpha) \in B} \mathbf{Pi}((p, \alpha))$
- The bridge between the two semantics:

  Proposition : $\mathbf{Pi}(B) = \{\pi : \pi(\omega) \leq \pi_B(\omega), \forall \omega \in \Omega\}$

  π_B is the least specific possibility distribution satisfying B.

Note that, while a possible world satisfies (p, α) to a degree, an epistemic state π satisfies it or not.

Beyond the conjunction connective : disjunction

- The conjunction of poslog formulas is captured by both semantics:

$$\mathbf{Pi}((p,\alpha) \wedge (q,\beta)) = \mathbf{Pi}((p,\alpha)) \cap \mathbf{Pi}((q,\beta)) = \{\pi | \pi \leq \min(\pi_{(p,\alpha)}, \pi_{(q,\beta)})\}.$$

- A disjunction of poslog formula is no longer a poslog formula, because

$$\mathbf{Pi}((p,\alpha) \vee (q,\beta)) = \{\pi | \; \pi_{(p,\alpha)} \geq \pi \text{ or } \pi_{(q,\beta)} \geq \pi\} = \mathbf{Pi}((p,\alpha)) \cup \mathbf{Pi}((q,\beta))$$

  no longer possesses a least specific element

- $(p,\alpha) \vee (q,\alpha)$ semantically differs from $(p \vee q, \alpha)$ since

$$\mathbf{Pi}((p \vee q, \alpha)) = \{\pi | \pi \leq \max(\pi_{(p,\alpha)}, \pi_{(q,\alpha)})\} \supseteq \mathbf{Pi}((p,\alpha) \cup \mathbf{Pi}((q,\alpha))$$

Only the epistemic semantics can account for disjunction of poslog formulas.

Beyond the conjunction connective : negation

- The negation $\neg(p, \alpha)$ of a poslog formula is no longer a poslog formula, because

$$\mathbf{Pi}(\neg(p, \alpha)) = \{\pi | \pi \not\leq \pi_{(p,\alpha)}\} = \overline{\mathbf{Pi}((p, \alpha))} \supset \mathbf{Pi}((\neg p, \alpha)).$$

- Again, $\neg(p, \alpha)$ has no ontic semantics since $\mathbf{Pi}(\neg(p, \alpha))$ has no greatest element.
- At the epistemic semantic level, it is clear that
  $\neg((p, \alpha) \wedge (q, \beta)) \equiv \neg(p, \alpha) \vee \neg(q, \beta)$
- To generalize poslog with disjunction and conjunction of poslog formulas one must drop the minimal specificity semantics and adopt the epistemic semantics.

Generalized possibilistic logic

- **Syntax** : Generalized possibilistic logic formulas are
  - Atoms are pairs (p, α) where p is a propositional formula and $\alpha \in L$.
  - A conjunction of formulas is a formula.
  - A disjunction of formulas is a formula.
  - The negation of a formula is a formula.
- **Semantic inference** :
  if Φ and Ψ are generalized poslog formulae, then $\Phi \models \Psi$ if and only if $\mathbf{Pi}(\Phi) \subseteq \mathbf{Pi}(\Psi)$.
  $B_{gen} \models \Psi$ iff $\cap_{\Phi \in B_{gen}} \mathbf{Pi}(\Phi) \subseteq \mathbf{Pi}(\Psi)$
- *Inference rule : Modus ponens* : $\Phi, \neg\Phi \vee \Psi \vdash \Psi$.

Possibilistic logic vs. generalised poslog : example

The difference between the formulas $(\neg p \vee q, \alpha)$ and $\neg(p, \alpha) \vee (q, \alpha), \alpha > 0$, in the presence of (p, α) affects inferences one may draw from them

- $(\neg p \vee q, \alpha); (p, \alpha) \vdash (q, \alpha)$ and $(\neg p \vee q, \alpha); (\neg q, \alpha) \vdash (\neg p, \alpha)$ hold ($N(\neg p) \geq \alpha$).
- $\neg(p, \alpha) \vee (q, \alpha); (p, \alpha) \vdash (q, \alpha)$ still holds
  but $\neg(p, \alpha) \vee (q, \alpha); (\neg q, \alpha) \vdash \neg(p, \alpha)$ only $(N(p) < \alpha)$.

Besides,

$\models (\neg p \vee q, \alpha) \to ((p, \alpha) \to (q, \alpha))$ (= $\neg(\neg p \vee q, \alpha) \vee \neg(p, \alpha) \vee (q, \alpha)$) holds:

it just says: if $N(\neg p \vee q) \geq \alpha$ and $N(p) \geq \alpha$ then $N(q) \geq \alpha$...

This is a weighted extension of axiom K.

Syntax for weighted epistemic formulas

A classical propositional language $\mathcal{L}$

Let $\Lambda = \{0, \frac{1}{k}, \frac{2}{k}, ..., 1\}$, where $k \in \mathbb{N} \setminus \{0\}$, the set of considered certainty levels

Idea encapsulate each formula α of $\mathcal{L}$ in a *valued* modality denoted $N_a(\alpha), a > 0$.

$$\text{possibility: } \Pi_b(\neg\alpha) := \neg N_a(\alpha), a + b = 1 - \frac{1}{k}.$$

- $N_a(\alpha)$ encodes constraint $N([\alpha]) \geq a$ for $a > 0$: previously denoted (α, a)
- $\Pi_b(\alpha)$ encodes constraint $\Pi([\alpha]) \geq b$ for $b > 0$
- $\neg N_a(\alpha)$ thus encodes $\Pi([\neg\alpha]) > 1 - a$, then $\Pi([\neg\alpha]) \geq 1 - a + \frac{1}{k}$, i.e. $\Pi_{1-a+\frac{1}{k}}(\neg\alpha)$
- we need at least 3 certainty levels ($k \geq 2$) in order to be able to distinguish between $\neg N_1(\alpha)$ and $\Pi_1(\neg\alpha)$.

$L\Pi G$: Axioms

- (LP)
- (K) : $N_a(\alpha \to \beta) \to (N_a(\alpha) \to N_a(\beta))$;
- (N) : $N_1(\alpha), \forall \alpha$ tel que $\vdash_{LP} \alpha$;
- (D) : $N_a(\alpha) \to \Pi_1(\alpha), \forall a > 0$;
- (AF) : $N_{a_1}(\alpha) \to N_{a_2}(\alpha)$, si $a_1 \geq a_2$.

Inference rule: (MP) $\{\phi, \phi \to \psi\} \vdash \psi$.

One recover the possibilistic logic modus ponens and the hybrid rule

- $\{N_{a_1}(\alpha), N_{a_2}(\alpha \to \beta)\} \vdash N_{\min(a_1,a_2)}(\beta)$
- $\{\Pi_{a_1}(\alpha), N_{a_2}(\alpha \to \beta)\} \vdash \Pi_{a_1}(\beta)$ si $a_2 > 1 - a_1$

The set of models of a formula ϕ in $L\Pi G$ is a set of possibility distributions π

Semantics

The satisfaction of formulas in $L\Pi G$ by possibility distributions is defined recursively:

- $\pi \models N_a(\alpha)$, iff $N([\alpha]) = \inf_{w \models \neg\alpha} 1 - \pi(w) \geq a, \forall \alpha \in \mathcal{L}$.
- $\pi \models \neg\phi$, iff $\pi \not\models \phi$.
- $\pi \models \phi \wedge \psi$, iff $\pi \models \phi$ and $\pi \models \psi$.

Let $\mathcal{B}$ be a base, the semantical inference $\mathcal{B} \models \phi$ means :

$$\forall \pi, \text{ if } \pi \models \psi, \forall \psi \in \mathcal{B} \text{ then } \pi \models \phi.$$

Compleness

Compleness Theorem $\mathcal{B} \vdash_{L\Pi G} \phi \iff \mathcal{B} \models_{L\Pi G} \phi$.

- As in propositional logic, $L\Pi G$ is sound and complete for its classical interpretations
- A propositional interpretation of the language $L\Pi G$ $v : \{N_a(\alpha), \alpha \in \mathcal{L}, a \in \Lambda \setminus \{0\}\} \to \{0, 1\}$ that satisfies (AF) is a set function:

$$g_v([\alpha]) = \max\{a : v(N_a(\alpha)) = 1\}.$$

- If v satisfies K, N, D then $g_v(\mathcal{V}) = 1$, $g_v(\emptyset) = 0$ and $g_v([\alpha \wedge \beta]) = \min(g_v([\alpha], g_v([\beta]))$.
- g_v is a necessity measure based on a unique possibility distribution π_v.

Thus classical interpretations of $L\Pi G$ are in a one-to-one correspondence with the possibility distributions.