State-of-the-Art on Reciprocal Relations

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SSIU 2014 & WIUI 2014, Olomouc, Czech Republic

April 10–11, 2014

Contents

1. Intransitivity of indifference

The Sorites Paradox

Many versions of the Sorites Paradox:

- **The Bald Man Paradox**: there is no particular number of hairs whose loss marks the transition to boldness

- **The Heap Paradox**: no grain of wheat can be identified as making the difference between a heap and not being a heap

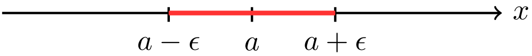
- **The Luce Paradox**: sugar in coffee example

The Poincaré Paradox

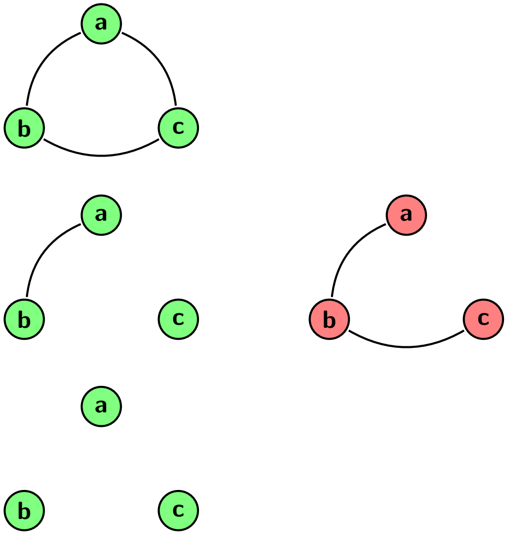
Approximate equality of real numbers is not transitive, i.e. stating that $a \in \mathbb{R}$ is similar to $b \in \mathbb{R}$ if

$$|a - b| \leq \epsilon$$

is **not transitive**



Possible symmetric configurations ($n = 3$)

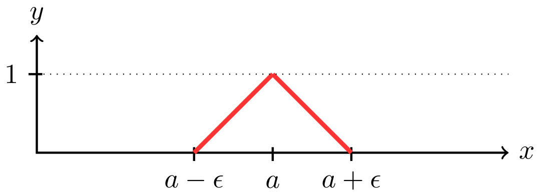


The Poincaré Paradox revisited

The fuzzy relation

$$E_\epsilon(a, b) = \max\left(1 - \frac{|a - b|}{\epsilon}, 0\right)$$

is $T_\mathbf{L}$**-transitive**, i.e. $E_\epsilon(a, b) + E_\epsilon(b, c) - 1 \leq E_\epsilon(a, c)$



The function $d_\epsilon = 1 - E_\epsilon$ is a metric: the **triangle inequality** holds

$$d_\epsilon(a, b) + d_\epsilon(b, c) \geq d_\epsilon(a, c)$$

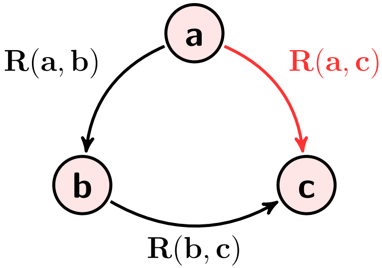
T-Transitivity of fuzzy relations

Fuzzy relation: $R : A^2 \rightarrow [0,1]$, with a **unipolar** semantics

- A fuzzy relation R on A is called **T-transitive**, with T a t-norm, if

$$T(R(a,b), R(b,c)) \leq R(a,c)$$

for any a, b, c in A



Triangular norms

Basic continuous t-norms:

minimum	$T_{\mathbf{M}}$	$\min(x, y)$
product	$T_{\mathbf{P}}$	xy
Łukasiewicz t-norm	$T_{\mathbf{L}}$	$\max(x + y - 1, 0)$

T-triplets

Consider three elements a_1, a_2 and a_3:

- A permutation (a_i, a_j, a_k) is called a **T-triplet** if

$$T(R(a_i, a_j), R(a_j, a_k)) \leq R(a_i, a_k)$$

- There can be at most 6 T-triplets
- T-transitivity expresses that there always are 6 T-triplets

2. Intransitivity of preference

Transitivity of preference

Transitivity of preference is a fundamental principle underlying most major rational, prescriptive and descriptive contemporary models of decision making

- **Rationality of individual and collective choice**: a transitive person, group or society that prefers choice option x to y and y to z must prefer x to z
- **Intransitive relations** are often perceived as something **paradoxical** and are associated with **irrational behaviour**
- Main argument: **money pump**

Intransitivity of preference

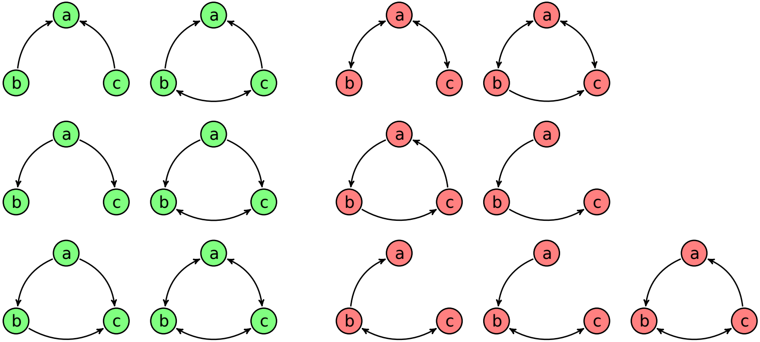
- **Transitivity** is expected to hold if preferences are based on a single scale (fitness maximization)
- **Intransitive choices** have been reported from both humans and other animals, such as **gray jays** (Waite, 2001) collecting food for storage

- **Bounded rationality**: intransitive choices are a suboptimal byproduct of heuristics that usually perform well in real-world situations (Kahneman and Tversky, 1969)
- Intransitive choices can result from decision strategies that maximize fitness (Houston, McNamara and Steer, 2007), as a kind of insurance against a run of bad luck

Intransitivity in life

Life provides many examples of intransitive relations, they often seem to be necessary and play a positive role

- sports: team A which defeated team B, which in turn won from C, can be overcome by C
- 13 love triangles:



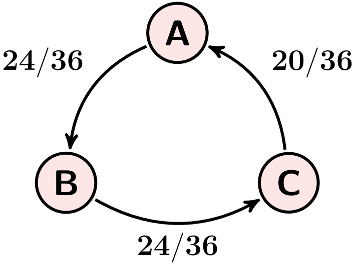
The God-Einstein-Oppenheimer dice puzzle

(New York Times, 30-03-09)

Integers 1–18 distributed over **3 dice**:

A	1	2	13	14	15	16
B	7	8	9	10	11	12
C	3	4	5	6	17	18

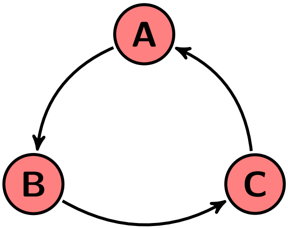
Winning probabilities:



Statistical preference

Statistical preference: X is preferred to Y if $\text{Prob}\{X > Y\} > \frac{1}{2}$

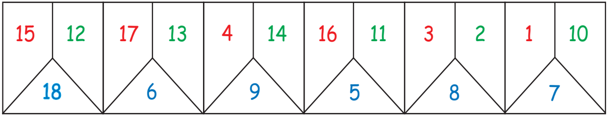
- May lead to cycles (Steinhaus and Trybuła, 1959):



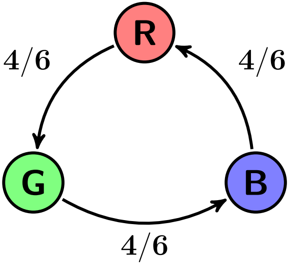
- There exist 10.705 cyclic distributions of the numbers 1–18 and 15 of them constitute a cycle of the highest equal probability $21/36 = 7/12$

A single die variant

Integers 1–18 distributed over **1 die**: 3 numbers on each face



Winning probabilities:

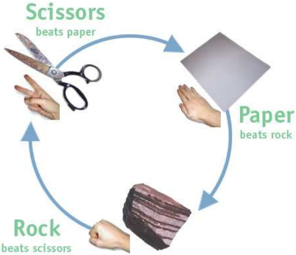


The single die can be seen as 3 coupled dice

Rock-Paper-Scissors

Cyclic dice are a type of **Rock-Paper-Scissors** (RPS):
(ancient children's game, *jan-ken-pon*, *rochambeau*)

- rock defeats scissors
- scissors defeat paper
- rock loses to paper



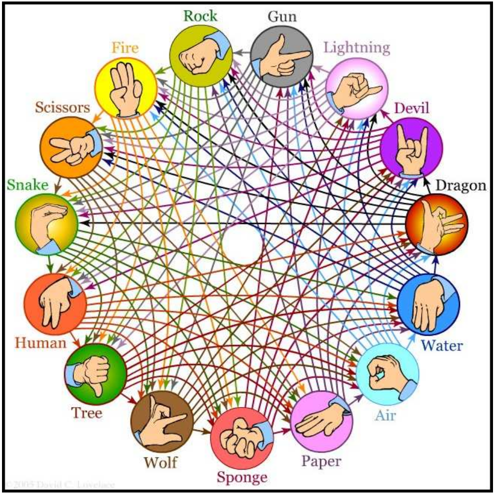
Rock-Paper-Scissors

The Rock-Paper-Scissors game:

- is often used as a **selection method** in a way similar to coin flipping, drawing straws, or throwing dice
- unlike truly random selection methods, RPS can be played with a **degree of skill**: recognize and exploit the non-random behaviour of an opponent
- **World RPS Society**:

  "Serving the needs of decision makers since 1918"

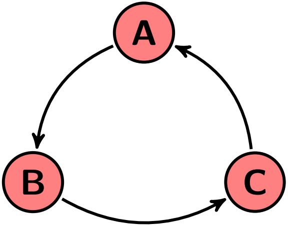
Rock-Paper-Scissors



RPS in voting

The voting paradox of Condorcet (Marquis de Condorcet, 1785)

voter 1:	A	$>$	B	$>$	C
voter 2:	B	$>$	C	$>$	A
voter 3:	C	$>$	A	$>$	B



Inspiration to **Arrow's impossibility theorem**: there is no choice procedure meeting the democratic assumptions

RPS in evolutionary biology: lizards

Common side-blotched **lizard** mating strategies (Sinervo and Lively, Nature, 1996) depending on the colour of throats of males

RPS in evolutionary biology: lizards

Lizard mating strategies:

- **orange beats blue**: males with orange throats can take territory from blue-throated males because they have more testosterone and body mass. As a result, orange males control large territories containing many females
- **blue beats yellow**: blue-throated males cooperate with each other to defend territories and closely guard females, so they are able to beat the sneaking strategy of yellow-throated males
- **yellow beats orange**: yellow-throated males are not territorial, but mimic female behavior and coloration to sneak onto the large territories of orange males to mate with females

RPS in evolutionary biology: Survival of the Weakest

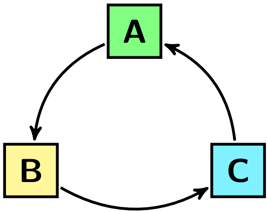
Cyclic competitions in spatial ecosystems (Reichenbach et al., 2007; Frey, 2009) (alternative to Lotka-Volterra equations, computer simulations using cellular automata)

- in large populations, the weakest species would - with very high probability - come out as the victor
- biodiversity in RPS games is negatively correlated with the rate of migration: critical rate of migration ϵ_{crit} above which biodiversity gets lost

Simulating microbial competition

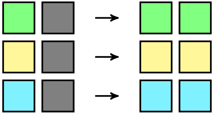
Simulation setting:

- three subpopulations: A, B, C
- initial population density: 25 % A, 25 % B, 25 % C, 25 %
- cellular automaton on a square grid
- environmental conditions discarded

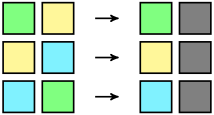


Simulating microbial competition: mechanisms

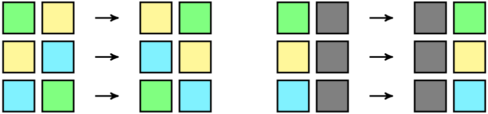
- Reproduction (μ):



- Selection (σ):



- Migration (ϵ):



Simulation experiment 1

$\epsilon < \epsilon_c$

Simulation experiment 2

$\epsilon > \epsilon_c$

3. Reciprocal relations

Reciprocal relations

Reciprocal relation: $Q : A^2 \to [0,1]$, with a **bipolar** semantics, satisfying

$$Q(a,b) + Q(b,a) = 1$$

- Example 1: **3-valued representation** of a **complete** relation R

$$Q(a,b) = \begin{cases} 1 & \text{, if } R(a,b) = 1 \text{ and } R(b,a) = 0 \\ 1/2 & \text{, if } R(a,b) = R(b,a) = 1 \\ 0 & \text{, if } R(a,b) = 0 \text{ and } R(b,a) = 1 \end{cases}$$

- Example 2: **winning probabilities** associated with a random vector $(X_1, X_2, \ldots, X_n)$

$$Q(X_i, X_j) = \mathrm{Prob}\{X_i > X_j\} + \frac{1}{2}\,\mathrm{Prob}\{X_i = X_j\}$$

Reciprocal relations

- Example 3: popular definition of a **"fuzzy" preference relation**

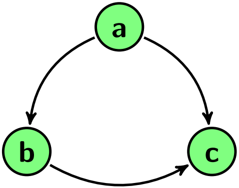
$$Q(a,b) = \begin{cases} \in \,]1/2, 1] & \text{, if } a \text{ is rather preferred to } b \\ 1/2 & \text{, if } a \text{ and } b \text{ are indifferent} \\ \in [0, 1/2[& \text{, if } b \text{ is rather preferred to } a \end{cases}$$

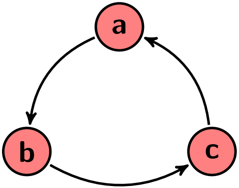
obeying the constraint $Q(a,b) + Q(b,a) = 1$, providing it with a **bipolar** semantics

Strong reservations against use of the word "fuzzy"

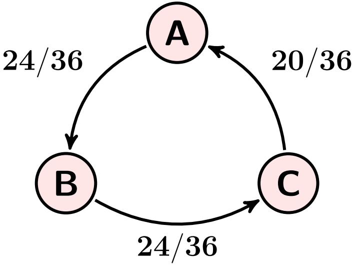
- **Bipolar** semantics
- Intersection makes no sense
  (cfr. intersection of complete relations is not complete)
- Fuzzy preference structures are more expressive

Possible complete asymmetric configurations ($n = 3$)





Oppenheimer's set of dice



Reciprocal relation:

$$Q = \begin{pmatrix} 1/2 & 24/36 & 16/36 \\ 12/36 & 1/2 & 24/36 \\ 20/36 & 12/36 & 1/2 \end{pmatrix}$$

Stochastic transitivity

A reciprocal relation Q is called **g-stochastic transitive** if

$$(Q(a,b) \geq 1/2 \ \wedge \ Q(b,c) \geq 1/2) \Rightarrow g(Q(a,b), Q(b,c)) \leq Q(a,c)$$

- **weak** stochastic transitivity ($g = 1/2$): iff 1/2-cut of Q is transitive
- **moderate** stochastic transitivity ($g = \min$):
  iff all α-cuts (with $\alpha \geq 1/2$) are transitive
- **strong** stochastic transitivity ($g = \max$)

A reciprocal relation Q is called **partially stochastic transitive** if

$$(Q(a,b) > 1/2 \ \wedge \ Q(b,c) > 1/2) \Rightarrow \min(Q(a,b), Q(b,c)) \leq Q(a,c) \ ;$$

iff all α-cuts (with $\alpha > 1/2$) are transitive

Isostochastic transitivity

A reciprocal relation Q is called ***h*-isostochastic transitive** if

$$(Q(a,b) \geq 1/2 \ \wedge \ Q(b,c) \geq 1/2) \ \Rightarrow \ h(Q(a,b), Q(b,c)) = Q(a,c)$$

- A reciprocal relation Q is called **multiplicatively transitive** (Tanino) if

$$\frac{Q(a,c)}{Q(c,a)} = \frac{Q(a,b)}{Q(b,a)} \cdot \frac{Q(b,c)}{Q(c,b)}$$

- Multiplicative transitivity = h-isostochastic transitivity w.r.t.

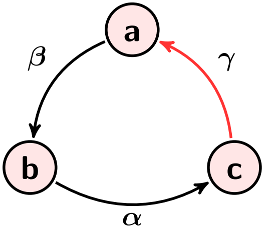
$$h(x,y) = \frac{xy}{xy + (1-x)(1-y)}$$

(Hamacher t-conorm of the 3Π-uninorm)

Cycle-transitivity

Reciprocal relation Q:

α_{abc}	$\min\{Q(a,b), Q(b,c), Q(c,a)\}$
β_{abc}	$\text{median}\{Q(a,b), Q(b,c), Q(c,a)\}$
γ_{abc}	$\max\{Q(a,b), Q(b,c), Q(c,a)\}$



Cycle-transitivity

- A reciprocal relation Q is called **cycle-transitive** w.r.t. an upper bound function U if

$$L(\alpha_{abc},\beta_{abc},\gamma_{abc}) \leq \alpha_{abc}+\beta_{abc}+\gamma_{abc}-1 \leq U(\alpha_{abc},\beta_{abc},\gamma_{abc})$$

- A function $U : \Delta = \{(x,y,z) \in [0,1]^3 \mid x \leq y \leq z\} \to \mathbb{R}$ is called an **upper bound function** if it satisfies:
  - $U(0,0,1) \geq 0$ and $U(0,1,1) \geq 1$
  - for any $(\alpha,\beta,\gamma) \in \Delta$:

$$U(\alpha,\beta,\gamma) \geq 1 - U(1-\gamma,1-\beta,1-\alpha)$$

- **Dual lower bound function**: function $L : \Delta \to \mathbb{R}$ defined by

$$L(\alpha,\beta,\gamma) = 1 - U(1-\gamma,1-\beta,1-\alpha)$$

Stochastic transitivity

- g-stochastic transitivity = cycle-transitivity w.r.t.

$$U_g(\alpha,\beta,\gamma) = \begin{cases} \boxed{\beta+\gamma-g(\beta,\gamma)} & \text{, if } \beta \geq 1/2 \wedge \alpha < 1/2 \\ 1/2 & \text{, if } \alpha \geq 1/2 \\ 2 & \text{, if } \beta < 1/2 \end{cases}$$

type	upper bound function	equivalent
weak	$\beta+\gamma-1/2$	
moderate	γ	
strong	β	β , if $\beta \geq 1/2$

Stochastic transitivity

- Partial stochastic trans. = cycle-trans. w.r.t. $U_{\rm ps}(\alpha, \beta, \gamma) = \gamma$:

$$\alpha_{abc} + \beta_{abc} \leq 1$$

- Multiplicative transitivity = cycle-transitivity w.r.t.

$$U_E(\alpha, \beta, \gamma) = \alpha\beta + \alpha\gamma + \beta\gamma - 2\alpha\beta\gamma$$

T-transitivity of reciprocal relations

Although not compatible with the bipolar semantics, T-transitivity can be imposed formally

- **1-Lipschitz** T: $|T(x_1,y_1) - T(x_2,y_2)| \leq |x_1 - x_2| + |y_1 - y_2|$
- T-transitivity = cycle-transitivity w.r.t.

$$U_T(\alpha,\beta,\gamma) = \alpha + \beta - T(\alpha,\beta)$$

t-norm	upper bound function	equivalent
$T_{\mathbf{M}}$	$\max(\alpha,\beta)$	β
$T_{\mathbf{P}}$	$\alpha+\beta-\alpha\beta$	
$T_{\mathbf{L}}$	$\min(\alpha+\beta,1)$	1

- $T_{\mathbf{M}}$-trans. = cycle-trans. w.r.t. $U(\alpha,\beta,\gamma) = \beta$:

$$\alpha_{abc} + \gamma_{abc} \leq 1$$

T-transitivity of reciprocal relations

Theorem

Consider a reciprocal relation on a set of three elements:

- There are either **3**, **5** or **6** $T_{\mathbf{M}}$**-triplets**
- There are either **3**, **4**, **5** or **6** $T_{\mathbf{P}}$**-triplets**
- There are either **3** or **6** $T_{\mathbf{L}}$**-triplets**

A non-symmetric triangle inequality

$T_{\mathbf{L}}$-transitivity of a reciprocal relation = "triangle inequality":

$$Q(a,b) + Q(b,c) \geq Q(a,c)$$

Product-triplets

Three variants of $T_{\mathbf{P}}$-transitivity:

name	upper bound f.	equiv. condition	# product-triplets
strong	$\alpha + \beta - \alpha\beta$	$\alpha\beta \leq 1 - \gamma$	6
moderate	$\alpha + \gamma - \alpha\gamma$	$\alpha\gamma \leq 1 - \beta$	≥ 5
weak	$\beta + \gamma - \beta\gamma$	$\beta\gamma \leq 1 - \alpha$	≥ 4

4. Winning probability relations

$T_{\mathbf{L}}$-transitivity of winning probability relations

Theorem

The **winning probability relation** associated with any random vector is $T_{\mathbf{L}}$-transitive, i.e. it satisfies the **triangle inequality**

$$Q(a,b) + Q(b,c) \geq Q(a,c)$$

A probabilistic viewpoint

Three random variables X_1, X_2 and X_3:

$$\mathrm{Prob}\{X_1 > X_2 \wedge X_2 > X_3\} \leq \mathrm{Prob}\{X_1 > X_3\}$$

Even if they are independent, then not necessarily

$$\mathrm{Prob}\{X_1 > X_2\}\,\mathrm{Prob}\{X_2 > X_3\} \leq \mathrm{Prob}\{X_1 > X_3\}$$

How close are winning probabilities to being $T_{\mathbf{P}}$-transitive

$$Q(a,b)Q(b,c) \leq Q(a,c)\ ?$$

Oppenheimer's set of dice

Reciprocal relation:

$$Q = \begin{pmatrix} 1/2 & 24/36 & 16/36 \\ 12/36 & 1/2 & 24/36 \\ 20/36 & 12/36 & 1/2 \end{pmatrix}$$

Four product-triplets, the only conditions **not** fulfilled are

$$Q(b,c)Q(c,a) \leq Q(b,a) \quad \text{and} \quad Q(c,a)Q(a,b) \leq Q(c,b)$$

since

$$\frac{20}{36} \times \frac{24}{36} = \frac{12}{36} + \frac{1}{27} > \frac{12}{36}$$

Pairwise independent random variables

Theorem (characterization for $n = 3$ and rational numbers)

The **winning probability relation** $Q^{\mathbf{P}}$ associated with **pairwise independent** random variables is **weakly $T_{\mathbf{P}}$-transitive** (dice-transitive), i.e.

$$\beta\gamma \leq 1 - \alpha$$

(both clockwise and counter-clockwise)

Interpretation

The winning probability relation $Q^{\mathbf{P}}$ is **at least $\frac{4}{6} \times 100\%$ $T_{\mathbf{P}}$-transitive**

Some interesting numbers for 3 dice

	4 faces	5 faces	6 faces	7 faces
4 $T_{\mathbf{P}}$-triplets	8.66%	1.67%	0.325%	0.060%
5 $T_{\mathbf{P}}$-triplets	14.01%	7.98%	4.2 %	2.31 %
6 $T_{\mathbf{P}}$-triplets	85.90%	92.00%	95.8%	97.68%
total number	5.78E+03	1.26E+05	2.86E+06	6.65+07

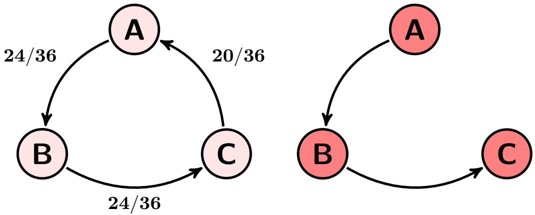
Exploiting dice-transitivity

- The relation $>^3_{\mathbf{P}}$:

$$X >^3_{\mathbf{P}} Y \quad \Leftrightarrow \quad Q^{\mathbf{P}}(X,Y) > \frac{\sqrt{5}-1}{2}$$

is an asymmetric relation **without cycles of length 3**

- The **golden section** $\phi = \frac{\sqrt{5}-1}{2}$: $\frac{22}{36} < \frac{\sqrt{5}-1}{2} < \frac{23}{36}$



Exploiting dice-transitivity

- The relation $>_{\mathbf{P}}^{k}$:

$$X >_{\mathbf{P}}^{k} Y \quad \Leftrightarrow \quad Q^{\mathbf{P}}(X,Y) > 1 - \frac{1}{4\cos^2(\pi/(k+2))}$$

  is an asymmetric relation **without cycles of length** k

- The relation $>_{\mathbf{P}}^{\infty}$:

$$X >_{\mathbf{P}}^{\infty} Y \quad \Leftrightarrow \quad Q^{\mathbf{P}}(X,Y) \geq \frac{3}{4}$$

  is an asymmetric **acyclic** relation

- The transitive closure $>_{\mathbf{P}}$ of $>_{\mathbf{P}}^{\infty}$ is a **strict order relation**

One- and two-parameter families

Marginal distributions belonging to a same parametric family:

- **One-parameter**: exponential, geometric, power-law (subfamilies of Beta and Pareto families), Gumbel

  multiplicative transitivity

- **Normal** distributions with same σ: ***h*-isostochastic transitivity** with

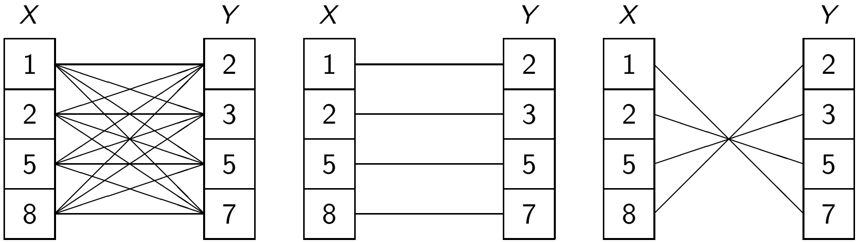
  $$h(x, y) = \Phi(\Phi^{-1}(x) + \Phi^{-1}(y))$$

  (with Φ the c.d.f. of standard normal distribution)

- **Normal** distributions:

  moderate stochastic transitivity

Independence - Co-monoton. - Counter-monoton.



$Q^{\mathbf{P}}(X, Y) = 7/16$ $\quad$ $Q^{\mathbf{M}}(X, Y) = 3/8$ $\quad$ $Q^{\mathbf{L}}(X, Y) = 1/2$

Copulas

- **Copula**: $C : [0,1]^2 \to [0,1]$ such that
  - neutral element 1, absorbing element 0
  - **2-increasingness**:

$$((x_1 \leq x_2 \wedge y_1 \leq y_2) \Rightarrow C(x_1, y_1) + C(x_2, y_2) \geq C(x_1, y_2) + C(x_2, y_1)$$

- Basic continuous t-norms are copulas and $T_{\mathbf{L}} \leq C \leq T_{\mathbf{M}}$
- Relationship between t-norms and copulas:

copula +	**associativity**	$\Rightarrow$	t-norm
t-norm +	**1-Lipschitz**	$\Rightarrow$	copula

- 1-Lipschitz t-norms = associative copulas

Sklar's theorem

- Sklar's theorem: for a random vector $(X_1, X_2, \ldots, X_n)$ there exist copulas C_{ij} s.t.

$$F_{X_i,X_j}(x,y) = C_{ij}(F_{X_i}(x), F_{X_j}(y))$$

- Captures dependence structure irrespective of the marginals

- Probabilistic interpretation:

$T_{\mathbf{M}}$	co-monotonicity
$T_{\mathbf{P}}$	independence
$T_{\mathbf{L}}$	counter-monotonicity

Dependence and the compatibility problem

- The **compatibility problem**:
  - not all combinations of copulas are possible
  - all $C_{ij} = C$ is possible for $C \in \{T_{\mathbf{M}}, T_{\mathbf{P}}\}$
  - $C_{12} = C_{13} = C_{23} = T_{\mathbf{L}}$ is impossible
- **Artificial coupling**:
  - winning probabilities require only bivariate coupling
  - **copula = comparison strategy**
  - does not (necessarily) reflect the real dependence

Extreme couplings

Choose a copula C as comparison strategy and compute the winning probabilities

$$Q^C(X,Y) = \mathrm{Prob}\{X > Y\} + \tfrac{1}{2}\,\mathrm{Prob}\{X = Y\}$$

Theorem

- The winning probabilities associated with random variables compared in a **co-monotone manner** satisfy the **triangle inequality**
- The winning probabilities associated with random variables compared in a **counter-monotone manner** satisfy **partial stochastic transitivity**

Exploiting cycle-transitivity: $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$

- The relation $>^k_{\mathbf{M}}$:

$$X >^k_{\mathbf{M}} Y \quad \Leftrightarrow \quad Q^{\mathbf{M}}(X,Y) > \frac{k-1}{k}$$

  is an asymmetric relation **without cycles of length k**

- The relation $>_{\mathbf{M}}$

$$X >_{\mathbf{M}} Y \quad \Leftrightarrow \quad Q^{\mathbf{M}}(X,Y) = 1$$

  is a **strict order relation**

- The relation $>_{\mathbf{L}}$

$$X >_{\mathbf{L}} Y \quad \Leftrightarrow \quad Q^{\mathbf{L}}(X,Y) > \frac{1}{2}$$

  is a **strict order relation**

The Frank copula family

- Frank family $(T_s^{\mathbf{F}})_{s\in[0,\infty]}$: for $s \in]0,1[\cup]1,\infty[$

$$T_s^{\mathbf{F}}(x,y) = \log_s\left(1 + \frac{(s^x-1)(s^y-1)}{s-1}\right)$$

- Limit cases:

0	$T_{\mathbf{M}}$
1	$T_{\mathbf{P}}$
∞	$T_{\mathbf{L}}$

- Prototypical solutions of the functional equation of Frank:

$$x + y - T(x,y) = 1 - T(1-x, 1-y)$$

- $T_s^{\mathbf{F}}$-transitivity = cycle-transitivity w.r.t.

$$U_s(\alpha,\beta,\gamma) = \alpha + \beta - T_s^{\mathbf{F}}(\alpha,\beta) = S_s^{\mathbf{F}}(\alpha,\beta)$$

Coupling by a Frank copula

Theorem

For a Frank copula $C = T_s^{\mathbf{F}}$, the reciprocal relation Q^C is cycle-transitive w.r.t.

$$U^C(\alpha, \beta, \gamma) = \beta + \gamma - T_{1/s}^{\mathbf{F}}(\beta, \gamma) = S_{1/s}^{\mathbf{F}}(\beta, \gamma)$$

copula	upper bound f.	equivalent	known as
$T_{\mathbf{M}}$	$\min(\beta + \gamma, 1)$	1	triangle inequality
$T_{\mathbf{P}}$	$\beta + \gamma - \beta\gamma$		dice-transitivity
$T_{\mathbf{L}}$	$\max(\beta, \gamma)$	γ	partial stoch. trans.

The Frank copula family

- Cutting levels:

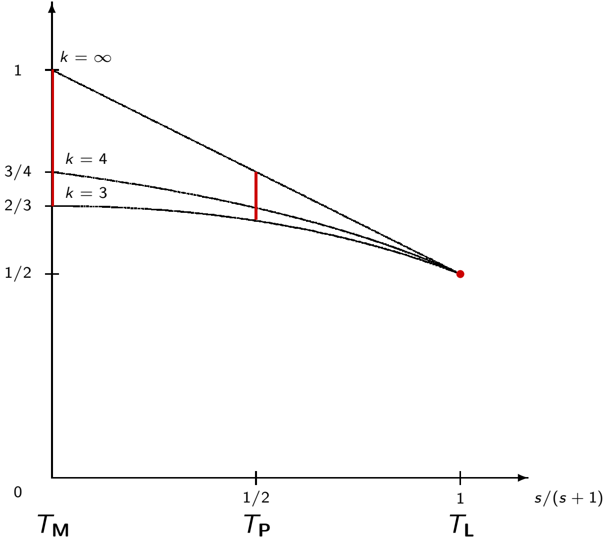
copula	s	level α_s
$T_{\mathbf{M}}$	0	$= 1$
$T_{\mathbf{P}}$	1	$\geq 3/4$
$T_{\mathbf{L}}$	∞	$> 1/2$

- The Frank copula family:

$$\alpha_s = 1 - \log_s \left(\frac{1 + \sqrt{s}}{2} \right)$$

$$\alpha_s + \alpha_{1/s} = 3/2$$

A picture says more than ...



5. Graded stochastic dominance

Stochastic dominance

Purpose of stochastic dominance:

- to define a **(partial) order relation** on a set of real-valued **random variables (RV)**
- should reflect that **RV taking higher values are preferred**

General principle:

- **pairwise comparison** of RV
- **pointwise comparison** of **performance functions** constructed from the **distribution function**

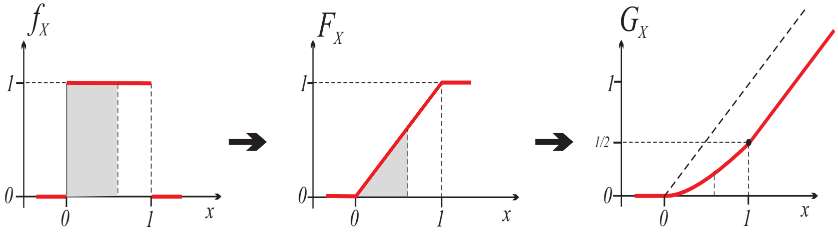
Performance functions

- The **cumulative distribution function** (CDF) F_X:

$$F_X(x) = \mathrm{Prob}\{X \leq x\}$$

- The **area below the CDF** F_X:

$$G_X(x) = \int_{-\infty}^{x} F_X(t)\,dt$$



1st and 2nd order stochastic dominance (SD)

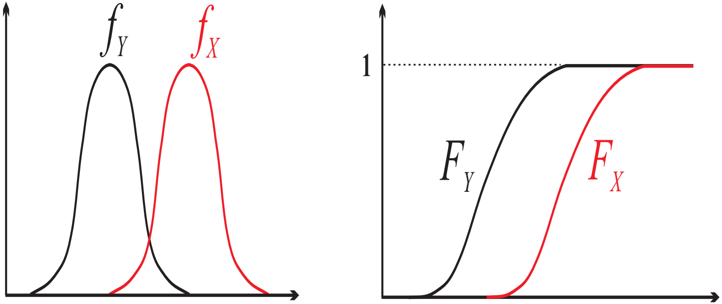
Stochastic dominance relation:

$X \succeq_{\mathrm{FSD}} Y$	$\stackrel{\mathrm{def}}{\Leftrightarrow}$	$F_X \leq F_Y$
	$\Leftrightarrow$	$\mathbf{E}[u(X)] \geq \mathbf{E}[u(Y)]$ for any **increasing** function u
$X \succeq_{\mathrm{SSD}} Y$	$\stackrel{\mathrm{def}}{\Leftrightarrow}$	$G_X \leq G_Y$
	$\Leftrightarrow$	$\mathbf{E}[u(X)] \geq \mathbf{E}[u(Y)]$ for any **increasing concave** function u

- Strict dominance relation:

$$X \succ Y \quad \Leftrightarrow \quad X \succeq Y \quad \text{and} \quad Y \not\succeq X$$

Graphical illustration of FSD



Application areas

- **Decision making under uncertainty**
- Risk averse preference models in **economics** and **finance**:
  - e.g. in portfolio optimisation
- **Social statistics**:
  - e.g. in the comparison of welfare and poverty indicators
- **Machine learning** and **multi-criteria decision making**:
  - e.g. in ranking (= ordered sorting) algorithms (OSDL, dominance-based rough sets, ...)

Discussion

- SD induces a **(classical) partial order relation** on a set of RV:
  - **no tolerance** for small deviations, **no grading**
  - partial: usually **sparse** graphs
- SD is theoretically attractive, but **computationally difficult**
- SD uses **marginal distributions** only
- SSD accumulates area from $-\infty$ onwards
  - introduces an **absolute reference point**

Main objective: graded variants of SD

- Our aim: construction of a reciprocal relation on a set of RV which allows to induce a strict order relation on the set of RV
- Choose a Frank copula $C = T_s^{\mathbf{F}}$ as comparison strategy and compute:

$$Q^C(X,Y) = \mathrm{Prob}\{X > Y\} + \tfrac{1}{2}\,\mathrm{Prob}\{X = Y\}$$

- The reciprocal relation Q^C is cycle-transitive w.r.t.

$$U^C(\alpha,\beta,\gamma) = \beta + \gamma - T^{\mathbf{F}}_{1/s}(\beta,\gamma)$$

- Compute (the transitive closure of) an appropriate (strict) α-cut of Q^C

Example: co-monotone comparison

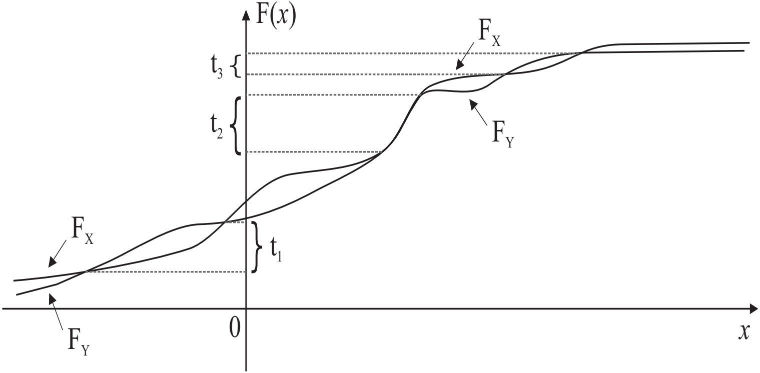
- The case of $T_{\mathbf{M}}$: continuous RV

$$Q^{\mathbf{M}}(X,Y) = \int_{x:F_X(x)<F_Y(x)} f_X(x)\,\mathrm{d}x + \frac{1}{2}\int_{x:F_X(x)=F_Y(x)} f_X(x)\,\mathrm{d}x$$

  - $Q^{\mathbf{M}}(X,Y) = 1$ iff $F_X < F_Y$ where $f_X \neq 0$:

more restrictive than $\succ_{\mathrm{FSD}}$

Graphical illustration



$$Q^{\mathbf{M}}(X, Y) = t_1 + t_3 + \frac{1}{2}\, t_2$$

Co-monotone comparison revisited

- The case of $T_{\mathbf{M}}$: discrete RV $$Q^{\mathbf{M}}(X,Y) = \frac{1}{n}\sum_{k=1}^{n} \delta_k^{\mathbf{M}}$$

  with

  $$\delta_k^{\mathbf{M}} = \begin{cases} 1 & \text{, if } x_k > y_k \\ 1/2 & \text{, if } x_k = y_k \\ 0 & \text{, if } x_k < y_k \end{cases}$$

- Parametrized version: $p \in \mathbb{R}^+$

  $$Q_p^{\mathbf{M}}(X,Y) = \frac{\displaystyle\sum_{k=1}^{n} (x_k - y_k)_+^p}{\displaystyle\sum_{k=1}^{n} |x_k - y_k|^p} = \frac{\mathbf{E}[(X-Y)_+^p]}{\mathbf{E}[|X-Y|^p]}$$

- Limit case: $Q_0^{\mathbf{M}} = Q^{\mathbf{M}}$

Co-monotone comparison revisited

- $p = 1$: **proportional expected difference**

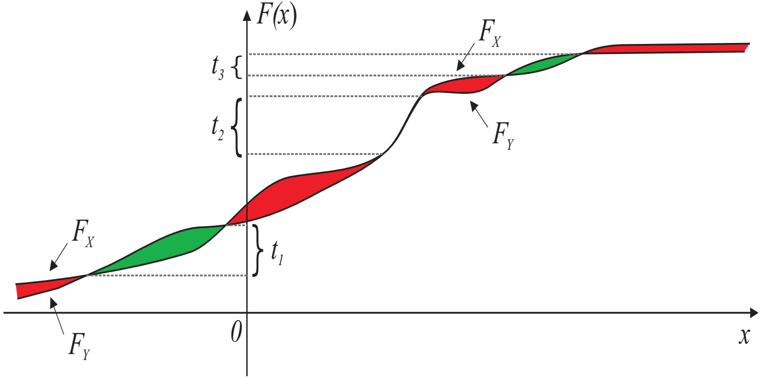
$$Q^{\mathrm{PED}}(X, Y) = \frac{\mathbf{E}[(X - Y)_+]}{\mathbf{E}[|X - Y|]}$$

with $Q^{\mathrm{PED}}(X, Y) = 1$ if and only if $X \succ_{\mathrm{FSD}} Y$

- The case of continuous RV and $p = 1$:

$$Q^{\mathrm{PED}}(X, Y) = \frac{\displaystyle\int (F_Y(x) - F_X(x))_+ \,\mathrm{d}x}{\displaystyle\int |F_Y(x) - F_X(x)| \,\mathrm{d}x}$$

Graphical illustration



Transitivity

Theorem

The proportional expected difference relation Q^{PED} is **partially stochastic transitive**

Use

- The strict 1/2-cut of Q^{PED} yields the strict order relation characterized by

$$Q^{\mathrm{PED}}(X,Y) > \frac{1}{2} \quad \Leftrightarrow \quad \mathbf{E}[X] > \mathbf{E}[Y]$$

- Any α-cut (with $\alpha > 1/2$) yields a **strict order relation**: with increasing α the graph (Hasse diagram) becomes more and more sparse (Hasse tree)

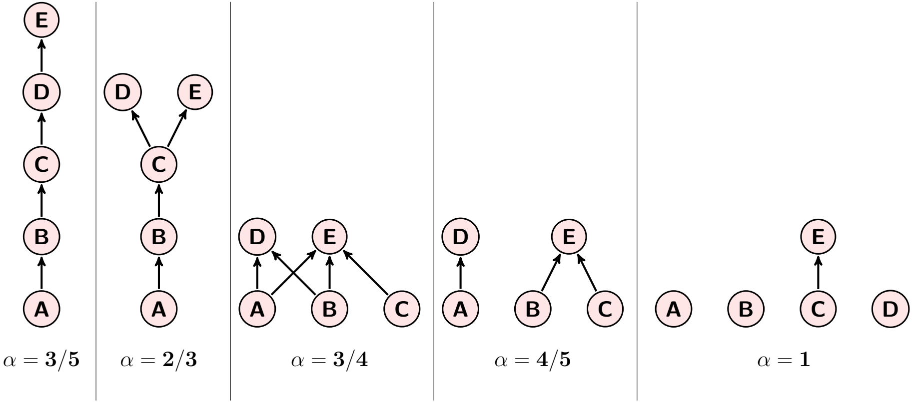
Example

Integers 1–9 distributed over **5 dice**:

A	1	4	9
B	3	4	8
C	3	6	7
D	2	7	8
E	5	6	7

$$Q^{\mathrm{PED}} = \begin{pmatrix} 1/2 & 1/3 & 1/3 & 1/5 & 1/4 \\ 2/3 & 1/2 & 1/3 & 1/4 & 1/5 \\ 2/3 & 2/3 & 1/2 & 1/3 & 0 \\ 4/5 & 3/4 & 2/3 & 1/2 & 2/5 \\ 3/4 & 4/5 & 1 & 3/5 & 1/2 \end{pmatrix}$$

Example



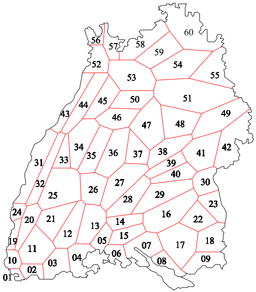
6. Poset ranking: coupled RV

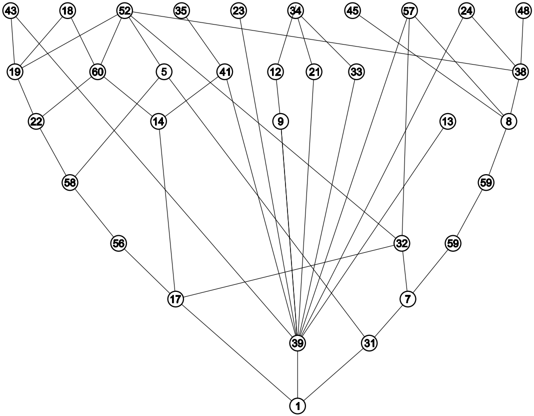
Partially ordered sets

Partially ordered sets (**posets**) are witnessing an increased interest:

- multi-criteria analysis without a common scale
- allow for incomparability
- usually based on product ordering in a multi-dimensional setting
- the **Hasse diagram technique** in **environmetrics** and **chemometrics**

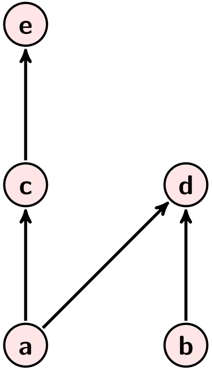
Real-world example: pollution in Baden-Württemberg

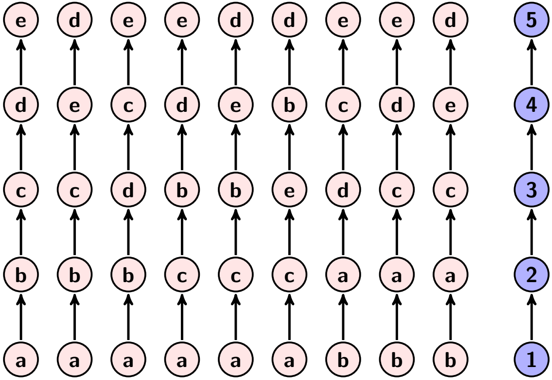




Toy example: a poset and its linear extensions

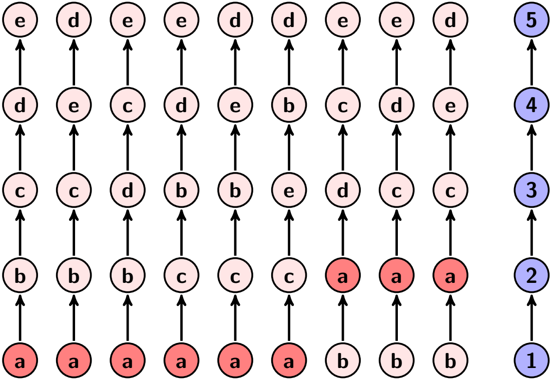
Linear extension: an **order-preserving permutation** of the elements

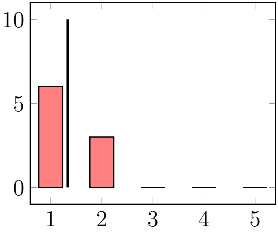




Toy example: average rank

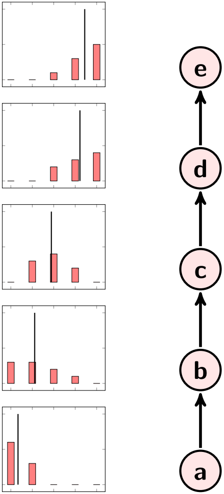
Discrete random variable X_a describing the position of a in a random linear extension





Toy example: poset ranking (weak order)

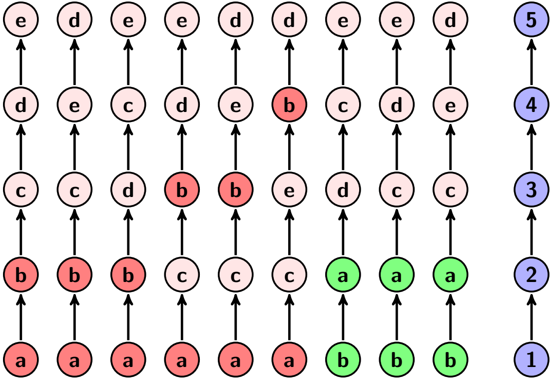
Ranking the elements according to their average rank $\rho(x_i) = \mathbf{E}[X_i]$



Toy example: mutual rank probabilities

Fraction of linear extensions in which *a* is ranked above *b*:

$$\text{Prob}\{X_a > X_b\} = \frac{3}{9}$$



Mutual rank probability relation

Mutual rank probability relation: reciprocal relation expressing the probability that x_i is ranked above x_j

$$Q_P(x_i, x_j) = \mathrm{Prob}\{X_i > X_j\}$$

Toy example:

$$Q = \begin{pmatrix} 1/2 & 3/9 & 0 & 0 & 0 \\ 6/9 & 1/2 & 3/9 & 0 & 1/9 \\ 1 & 6/9 & 1/2 & 2/9 & 0 \\ 1 & 1 & 7/9 & 1/2 & 4/9 \\ 1 & 8/9 & 1 & 5/9 & 1/2 \end{pmatrix}$$

Mutual rank probability relation

- Distribution of the random vector $(X_1, \ldots, X_n)$ depends on the structure of the poset (if x_i and x_j are comparable, then $C_{ij} = T_{\mathbf{M}}$)
- Average rank in terms of mutual rank probabilities:

$$\rho(x_i) = 1 + \sum_{j \neq i} Q_P(x_i, x_j)$$

- Proportional transitivity (Fishburn, 1986; Yu, 1998):

$$(Q_P(a,b) \geq u \ \wedge \ Q_P(b,c) \geq u) \ \Rightarrow \ Q_P(a,c) \geq u$$

holds for $u \geq \rho \approx 0.78$

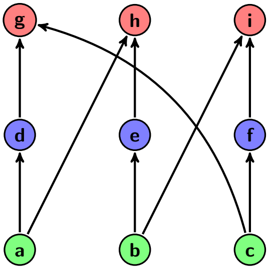
Linear extension majority cycles

The **Linear Extension Majority** (LEM) relation is the strict 1/2-cut of Q_P: x_i is ranked above x_j if

$$\mathrm{Prob}\{X_i > X_j\} > \frac{1}{2}$$

- The LEM relation may contain cycles (if $n \geq 9$): LEM k-cycles
- Only 5 out of 183 231 posets of size 9 contain LEM 3-cycles, none of them contains longer LEM cycles

Linear extension majority cycles



$$Q(g,h) = Q(h,i) = Q(i,g) = \frac{720}{1431}$$
$$Q(d,e) = Q(e,f) = Q(f,d) = \frac{720}{1431}$$
$$Q(a,b) = Q(b,c) = Q(c,a) = \frac{720}{1431}$$

- the strict α-cut at $\alpha = \dfrac{720}{1431} = 0.50314465$ is cycle-free
- only one poset of size 9 requires this α

Proportional transitivity in posets

- Find largest $\delta : [0,1]^2 \rightarrow [0,1]$ such that for any finite poset

$$\delta(Q_P(x_i, x_j), Q_P(x_j, x_k)) \leq Q_P(x_i, x_k)$$

- Kahn and Yu (1998): $\delta^* \leq \delta$ with δ^* the conjunctor

$$\delta^*(u,v) = \begin{cases} 0 & \text{, if } u+v<1 \\ \min(u,v) & \text{, if } u+v-1 \geq \min(u^2,v^2) \\ \dfrac{(1-u)(1-v)}{(1-\sqrt{u+v-1})^2} & \text{, elsewhere} \end{cases}$$

Transitivity

Theorem

The mutual rank probability relation is **moderately $T_{\mathbf{P}}$-transitive**, i.e.

$$\alpha\gamma \leq 1 - \beta$$

(both clockwise and counter-clockwise)

Interpretation

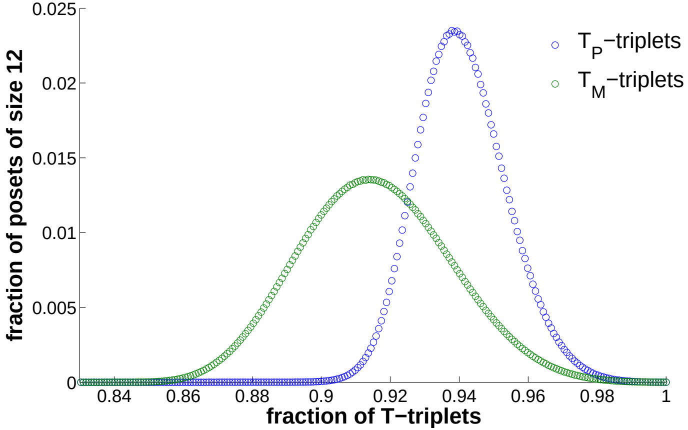
The mutual rank probability relation is **at least $\frac{5}{6} \times 100\%$ $T_{\mathbf{P}}$-transitive**

Avoiding 3-cycles

The strict ϕ-cut of Q_P, with $\phi = 0.618034$ the **golden section**, contains no cycles of length 3

Product-triplets and min-triplets

There are 1 104 891 746 non-isomorphic posets of 12 elements



7. Ranking representability

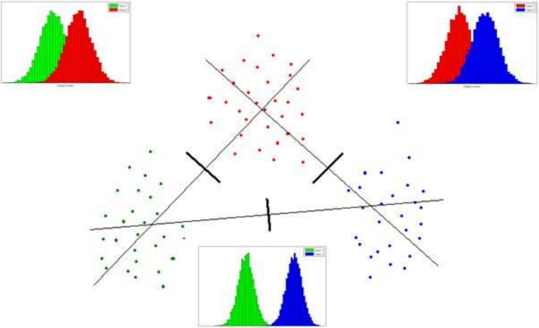
Machine learning setting

- Object space $\mathcal{X}$ (usually m-dimensional vector space) and a finite label set $\mathcal{L} = \{\lambda_1, \ldots, \lambda_r\}$
- Unknown distribution $\mathcal{D}$ over $\mathcal{X} \times \mathcal{L}$
- Conditional distributions $\mathcal{D}_j$
- I.i.d. data sample of size n: $D = \{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n)\}$
- **One-versus-one** method: $r(r-1)/2$ data subsamples

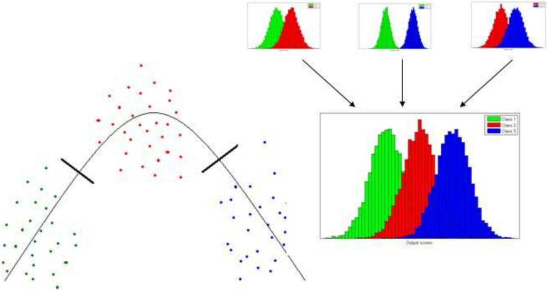
$$D_{kl} = \{(\mathbf{x}_i, y_i) \in D \mid y_i \in \{\lambda_k, \lambda_l\}\}$$

with $1 \leq k < l \leq r$

One-versus-one classification



Reduce MC classification to ordinal regression?



Binary classification

- Two classes labelled λ_k and λ_l (say $\lambda_k < \lambda_l$)
- Ranking function $f : \mathcal{X} \to \mathbb{R}$
- Performance evaluation: **AUC (area under the ROC curve)**

$$\hat{A}(f, D_{kl}) = \frac{1}{n_k n_l} \sum_{y_i < y_j} I_{\{f(\mathbf{x}_i) < f(\mathbf{x}_j)\}} + \frac{1}{2} I_{\{f(\mathbf{x}_i) = f(\mathbf{x}_j)\}}$$

  - Receiver Operating Characteristics
  - Mann-Whitney-Wilcoxon statistic
  - unbiased non-parametric estimator of the **Expected Ranking Accuracy (ERA)**

$$A_{kl}(f) = \mathrm{Prob}\{f(X_k) < f(X_l)\} + \frac{1}{2} \mathrm{Prob}\{f(X_k) = f(X_l)\}$$

  with $X_k \sim \mathcal{D}_k$ and $X_l \sim \mathcal{D}_l$

Strict ranking representability

One-versus-one: $r(r-1)/2$ ranking functions f_{kl} trained on data sets D_{kl}

Strict ranking representability

The ensemble $\{f_{kl}\}$ is called **strictly ranking representable** if there exists a ranking function $f : \mathcal{X} \to \mathbb{R}$ s.t. for all $1 \leq k < l \leq r$ and all $(\mathbf{x}_i, y_i), (\mathbf{x}_j, y_j) \in D_{kl}$

$$f_{kl}(\mathbf{x}_i) < f_{kl}(\mathbf{x}_j) \quad \Longleftrightarrow \quad f(\mathbf{x}_i) < f(\mathbf{x}_j)$$

[**Assumption**: pairwise ranking functions and the single ranking function have a similar degree of complexity]

Verifying strict ranking representability:

- algorithm linear in the size of the data set (topological sorting)
- limited applicability

AUC ranking representability

- Goal is a good performance on independent test data, not exactly the same result on some training data!
- Relaxation: require the same **performance** rather than the same results
- The ensemble $\{f_{kl}\}$ is **AUC ranking representable** if there exists a ranking function $f : \mathcal{X} \to \mathbb{R}$ s.t. for all $1 \leq k < l \leq r$

$$\hat{A}(f_{kl}, D_{kl}) = \hat{A}(f, D_{kl})$$

AUC ranking representability

- For $k < l$, add the ranking function $f_{lk} = -f_{kl}$
- The AUC form a reciprocal relation (put $Q(k,k) = \frac{1}{2}$)

$$Q(k,l) = \hat{A}(f_{kl}, D_{kl})$$

  - Strict ranking representability implies AUC ranking representability
  - AUC ranking representability implies dice-transitivity of Q, i.e. cycle-transitivity w.r.t.

$$U_D(\alpha, \beta, \gamma) = \beta + \gamma - \beta\gamma$$

  - $T_{\mathbf{M}}$-transitivity of Q does **NOT** imply AUC ranking representability

ERA ranking representability

- The ensemble $\{f_{kl}\}$ is **ERA ranking representable** if there exists a ranking function $f : \mathcal{X} \to \mathbb{R}$ s.t. for all $1 \leq k < l \leq r$

$$A_{kl}(f_{kl}) = A_{kl}(f)$$

- For $k < l$, add the ranking function $f_{lk} = -f_{kl}$
- The ERA form a reciprocal relation: $Q(k, l) = A_{kl}(f_{kl})$
- Three-class case ($r = 3$): the ensemble $\{f_{kl}\}$ is **ERA ranking representable** iff Q is κ-transitive with κ the conjunctor

$$\kappa(u, v) = \begin{cases} 0 & , \text{ if } u + v < 1 \\ uv & , \text{ if } u + v \geq 1 \end{cases}$$

  - Situated between dice-transitivity and $T_{\mathbf{P}}$-transitivity

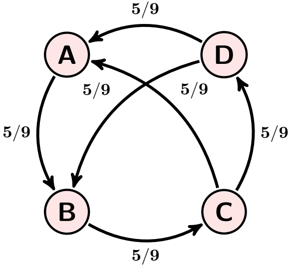
8. More dice games: beyond transitivity

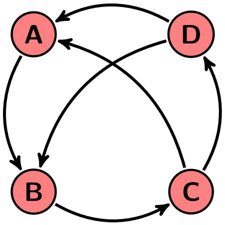
Rock-Paper-Scissors-Lizard

Integers 1–12 distributed over **4 dice**:

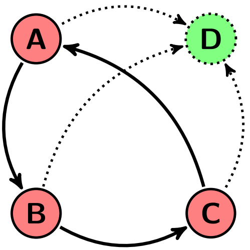
A	1	6	12
B	4	5	10
C	3	8	9
D	2	7	11

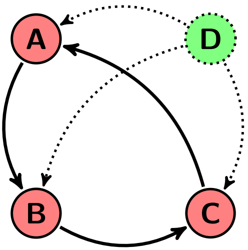
Statistical preference: 4-cycle $ABCD$ and two 3-cycles ABC and BCD

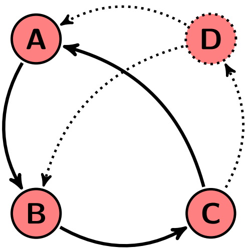


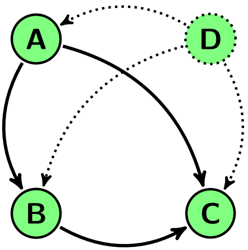


Possible complete asymmetric configurations ($n = 4$)









Product-triplets ($n = 4$)

Interpretation

The winning probability relation $Q^{\mathbf{P}}$ is **at least** $\frac{4}{6} \times 100\%$ $T_{\mathbf{P}}$**-transitive**

Some figures: number of product-triplets for 4 dice

	4 faces	5 faces	6 faces
16 triplets	-	-	-
17 triplets	-	-	0.000001 %
18 triplets	0.001%	0.00004%	0.000003 %
19 triplets	0.010%	0.0013%	0.0001%
20 triplets	0.26%	0.080%	0.018 %
21 triplets	3.37%	1.51%	0.54 %
22 triplets	17.45%	9.48%	4.91 %
23 triplets	10.63%	8.23%	5.35 %
24 triplets	68.28%	80.69%	89.18%
total number	2.63E+06	4.89E+08	9.30E+10

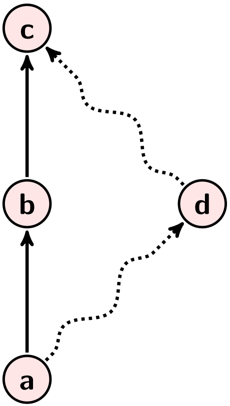
At least 16 product-triplets it is!

Integers 1–36 distributed over **4 dice**:

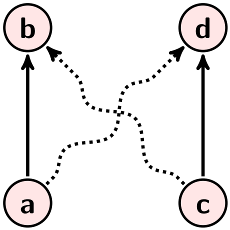
A	4	5	6	7	8	9	10	34	35
B	11	12	13	14	15	16	17	18	36
C	1	19	20	21	22	23	24	25	26
D	2	3	27	28	29	30	31	32	33

Semi-transitivity and the Ferrers property

Semi-transitivity:
if aRb and bRc, then aRd or dRc



The Ferrers property:
if aRb and cRd, then aRd or cRb



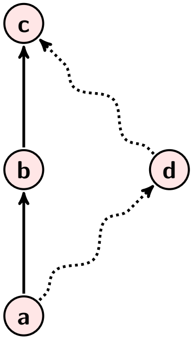
Key property of methods for **ranking fuzzy intervals (numbers)**, rather than transitivity!

T-semi-transitivity

A fuzzy relation R on A is called **T-semi-transitive**, with T a t-norm and T^* its dual t-conorm, if

$$T(R(a,b),R(b,c)) \leq T^*(R(a,d),R(d,c))$$

for any a, b, c, d in A

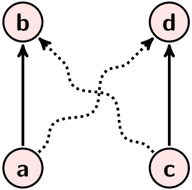


T-Ferrers property

A fuzzy relation R on A is called **T-Ferrers**, with T a t-norm and T^* its dual t-conorm, if

$$T(R(a,b),R(c,d)) \leq T^*(R(a,d),R(c,b))$$

for any a, b, c, d in A



Reciprocal relations

- **Complete relations**: transitivity implies semi-transitivity and the Ferrers property
- **Reciprocal relations**: if T is 1-Lipschitz continuous, then
  - T-transitivity implies T-semi-transitivity
  - T-transitivity implies the T-Ferrers property

$T_{\mathbf{L}}$-Ferrers

The **winning probability relation** associated with a random vector is $T_{\mathbf{L}}$-Ferrers

The Ferrers property

Four **independent** random variables X_1, X_2, X_3 and X_4:

$$\mathrm{Prob}\{X_1 > X_2\}\mathrm{Prob}\{X_3 > X_4\}$$

$$\leq \mathrm{Prob}\{X_1 > X_4\} + \mathrm{Prob}\{X_3 > X_2\} - \mathrm{Prob}\{X_1 > X_4\}\mathrm{Prob}\{X_3 > X_2\}$$

Theorem

The **winning probability relation** Q^{P} associated with pairwise independent random variables is **$T_{\mathbf{P}}$-Ferrers**

A stronger version of the $T_{\mathbf{P}}$-Ferrers property

Weak $T_{\mathbf{P}}$-transitivity and the $T_{\mathbf{P}}$-Ferrers property revisited

- A reciprocal relation Q is weakly $T_{\mathbf{P}}$-transitive (dice-transitive) if and only if for any 3 consecutive weights (t_1, t_2, t_3) it holds that

$$t_1 + t_2 + t_3 - 1 \geq \min(t_1 t_2, t_2 t_3, t_3 t_1)$$

- A reciprocal relation Q is $T_{\mathbf{P}}$-Ferrers if and only if for any 4 consecutive weights (t_1, t_2, t_3, t_4) it holds that

$$t_1 + t_2 + t_3 + t_4 - 1 \geq t_1 t_3 + t_2 t_4$$

4-cycle condition

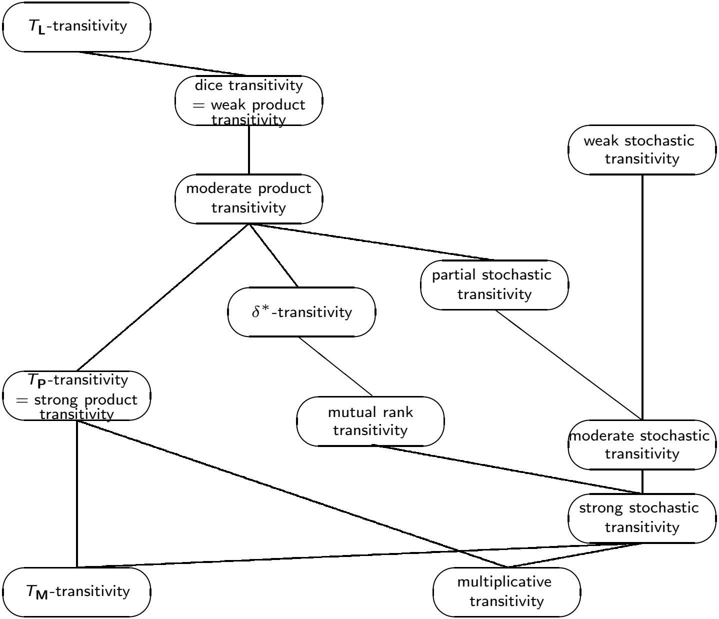
The winning probability relation $Q^{\mathbf{P}}$ associated with pairwise independent random variables satisfies for any for any 4 consecutive weights (t_1, t_2, t_3, t_4)

$$t_1 + t_2 + t_3 + t_4 - 1 \geq t_1 t_3 + t_2 t_4 + \min(t_1, t_3) \min(t_2, t_4)$$

Conclusion

Conclusion

- **Cyclic phenomena** are not necessarily incompatible with transitivity, but arise due to the **granularity** considered
- **Cycle-transitivity** yields a general framework for studying the transitivity of **reciprocal relations**
- **Frequentist interpretation** of the transitivity of **winning probabilities** in terms of product-transitivity
- Alternative theories of **stochastic dominance**
- **AUC** as a means to distinguish between multi-class classification and ordinal regression
- **In silico species competition** and coexistence


$T_{\mathbf{L}}$-transitivity
dice transitivity = weak product transitivity
weak stochastic transitivity
moderate product transitivity
partial stochastic transitivity
δ^*-transitivity
$T_{\mathbf{P}}$-transitivity = strong product transitivity
mutual rank transitivity
moderate stochastic transitivity
strong stochastic transitivity
$T_{\mathbf{M}}$-transitivity
multiplicative transitivity

What if God does throw dice?

Integers 1–20 distributed over **5 dice**:

A	1	5	12	20
B	2	6	15	18
C	3	9	14	17
D	4	8	11	19
E	7	10	13	16

Whatever X, Y selected by Oppenheimer and Einstein, God can select Z such that

$$\mathrm{Prob}\{Z > \max(X,Y)\} > \mathrm{Prob}\{X > \max(Y,Z)\}$$

$$\mathrm{Prob}\{Z > \max(X,Y)\} > \mathrm{Prob}\{Y > \max(X,Z)\}$$

This cannot be realized with 3 or 4 dice

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