A Review - The Exercices

Esko Turunen

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1° Calculate Łukasiewicz product, sum, implication and negation for $a = 0.9, b = 0.8$.

Solution.
$0.9 \odot 0.8 = \max\{0.9 + 0.8 - 1, 0\} = 0.7$
$0.9 \oplus 0.8 = \min\{0.9 + 0.8, 1\} = 1$
$0.9^* = 1 - 0.9 = 0.1,\ 0.8^* = 1 - 0.8 = 0.2$
$0.9 \rightarrow 0.8 = \min\{1 - 0.9 + 0.8, 1\} = 0.9$

2° Prove that De Morgan law $a \odot b = (a^* \oplus b^*)^*$ holds for all $a, b \in [0, 1]$.

Solution.
$(a^* \oplus b^*)^* = 1 - \min\{1 - a + 1 - b, 1\} =$
$\max\{1 - 1 + a - 1 + b, 0\} = a \odot b$

3° Study the following generalized syllogism.

Many mushrooms are not tasty.
Almost–all mushrooms are edible natural issues.
———
Some edible natural issue are not tasty.

a) To which Figure does it belong and b) is it valid?

Solution.

$$\frac{\begin{array}{l}\text{MP - Many not} = \text{G}\\ \text{MS - Almost all} = \text{P}\end{array}}{\text{SP - Some not} = \text{O}}$$

(a) The Figure is III, the syllogism is GPO-III (b) which is a valid syllogism.

4° Let the truth value set an MV-algebra. Show that

(Ax.12) $[\alpha \texttt{ or } (\texttt{not}\, \alpha \texttt{ and } \beta)] \texttt{ imp } [(\alpha \texttt{ imp } \beta) \texttt{ imp } \beta]$,

where α, β are wffs obtain value **1** in all valuations.

Solution.
The claim is true iff $a \oplus (a^* \odot b) \leq (a \to b) \to b$ in any MV-algebra; this is the case as

$$
\begin{aligned}
(a \to b) \to b &= a \vee b \\
&= b \vee a \\
&= (b \to a) \to a \\
&= (b^* \oplus a)^* \oplus a \\
&= a \oplus (b^* \oplus a)^* \\
&= a \oplus (b^{**} \odot a^*) \\
&= a \oplus (b \odot a^*) \\
&= a \oplus (a^* \odot b).
\end{aligned}
$$

5° Let the truth value set an MV-algebra. Prove that Generalized Modus Tollendo Tollens

$$\frac{\texttt{not}\,\beta, \alpha\ \texttt{imp}\ \beta}{\texttt{not}\,\alpha} \quad , \quad \frac{a, b}{a \odot b}$$

is a fuzzy rule of inference in Pavelka's sense.

Solution. $r^{\texttt{sem}}$ is obviously isotone: is $b \leq c$ then $a \odot b \leq a \odot c$. To see that soundness holds we remark that
(i) $(a \to b) \odot (b \to c) \leq a \to c$ holds in MV-algebras. Now

$$\begin{aligned} r^{\texttt{sem}}(v(\texttt{not}\,\beta), v(\alpha\ \texttt{imp}\ \beta)) &= v(\texttt{not}\,\beta) \odot v(\alpha\ \texttt{imp}\ \beta) \\ &= (v(\alpha) \to v(\beta)) \odot (v(\beta) \to 0) \\ &\leq v(\alpha) \to 0 \\ &= v(\texttt{not}\,\alpha) \\ &= v(r^{\texttt{syn}}(\texttt{not}\,\beta, \alpha\ \texttt{imp}\ \beta)). \end{aligned}$$

6° Assume α and β are associated with evidence couples $\langle 0.9, 0.2\rangle$ and $\langle 0.6, 0.1\rangle$. What are the corresponding evidence matrices of α, β, $\texttt{not}\,\alpha$, $\texttt{not}\,\beta$, α $\texttt{and}$ β, α $\texttt{or}$ β and α $\texttt{imp}$ β?

Solution.

$$v(\alpha) = \begin{bmatrix} 0.9^* \wedge 0.2 & 0.9 \odot 0.2 \\ 0.9^* \odot 0.2^* & 0.9 \wedge 0.2^* \end{bmatrix} = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.8 \end{bmatrix}$$

$$v(\texttt{not}\,\alpha) = \begin{bmatrix} 0.8 & 0 \\ 0.1 & 0.1 \end{bmatrix}$$

$$v(\beta) = \begin{bmatrix} 0.6^* \wedge 0.1 & 0.6 \odot 0.1 \\ 0.6^* \odot 0.1^* & 0.6 \wedge 0.1^* \end{bmatrix} = \begin{bmatrix} 0.1 & 0 \\ 0.3 & 0.6 \end{bmatrix}$$

$$v(\texttt{not}\,\beta) = \begin{bmatrix} 0.6 & 0.3 \\ 0 & 0.1 \end{bmatrix}$$

For $v(\alpha \texttt{ or } \beta)$ it is easier to calculate first the evidence couple $\langle 0.9, 0.2\rangle \oplus \langle 0.6, 0.1\rangle = \langle 0.9 \oplus 0.6, 0.2 \odot 0.1\rangle = \langle 1, 0\rangle$.

$$\text{Therefore} \quad v(\alpha \texttt{ or } \beta) \quad = \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

For $v(\alpha \texttt{ and } \beta)$ we calculate the evidence couple $\langle 0.9, 0.2\rangle \odot \langle 0.6, 0.1\rangle = \langle 0.9 \odot 0.6, 0.2 \oplus 0.1\rangle = \langle 0.5, 0.3\rangle$.
Therefore

$$v(\alpha \texttt{ and } \beta) = \begin{bmatrix} 0.5^* \wedge 0.3 & 0.5 \odot 0.3 \\ 0.5^* \odot 0.3^* & 0.5 \wedge 0.3^* \end{bmatrix} = \begin{bmatrix} 0.3 & 0 \\ 0.2 & 0.5 \end{bmatrix}$$

For $v(\alpha \texttt{ imp } \beta)$ we calculate the evidence couple $\langle 0.9, 0.2\rangle \rightarrow \langle 0.6, 0.1\rangle = \langle 0.9 \rightarrow 0.6, (0.1 \rightarrow 0.2)^*\rangle = \langle 0.7, 0\rangle$.
Therefore

$$v(\alpha \texttt{ imp } \beta) = \begin{bmatrix} 0.7^* \wedge 0 & 0.7 \odot 0 \\ 0.7^* \odot 0^* & 0.7 \wedge 0^* \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0.3 & 0.7 \end{bmatrix}$$

To calculate the evidence matrix for (α equiv β) we first have to calculate the evidence couple for (β imp α), which is $\langle 0.6, 0.1\rangle \to \langle 0.9, 0.2\rangle = \langle 0.6 \to 0.9, (0.2 \to 0.1)^*\rangle = \langle 1, 0.1\rangle$.
Since (α equiv β) = (α imp β)$\overline{\text{and}}$(β imp α), the corresponding evidence matrix is defined by the evidence couple $\langle 0.7, 0\rangle \wedge \langle 1, 0.1\rangle$ $= \langle 0.7 \wedge 1, 0 \vee 0.1\rangle = \langle 0.7, 0.1\rangle$.
Therefore

$$v(\alpha \text{ imp } \beta) = \left[\begin{array}{ll} 0.7^* \wedge 0.1 & 0.7 \odot 0.1 \\ 0.7^* \odot 0.1^* & 0.7 \wedge 0.1^* \end{array}\right] = \left[\begin{array}{ll} 0.1 & 0 \\ 0.2 & 0.7 \end{array}\right]$$