GAUSSIAN INTRINSIC ENTANGLEMENT

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Motivation

Seek for computable meaningful Gaussian entanglement measure

Classical mutual information of a Gaussian state

$$I_c(\rho_{AB}^G) = \sup_{\Gamma_A,\Gamma_B} \frac{1}{2}\left\{\ln\left[\frac{\det(\gamma_A+\Gamma_A)\det(\gamma_B+\Gamma_B)}{\det(\gamma_{AB}+\Gamma_A\oplus\Gamma_B)}\right]\right\}$$

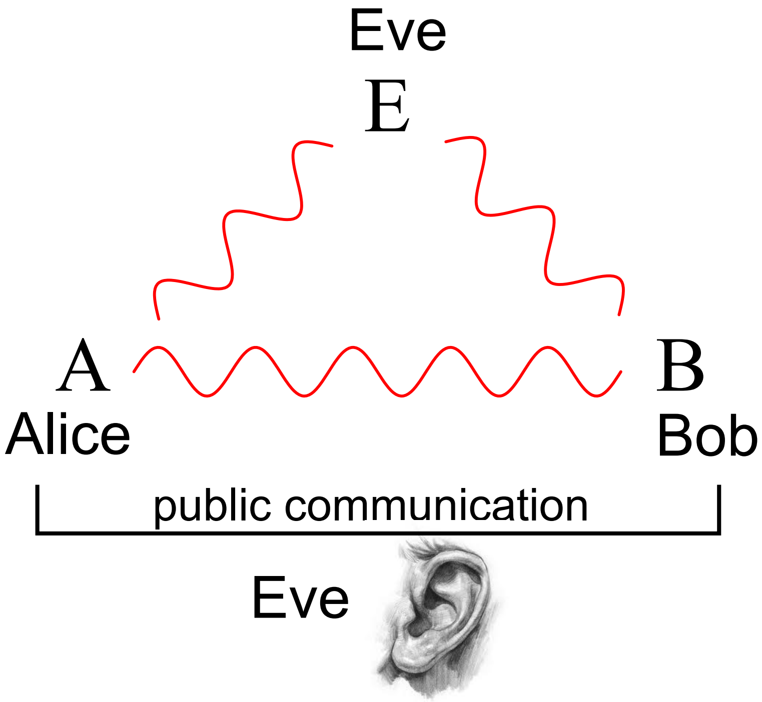
(L. Mista Jr *et al.*, Phys. Rev. A **83**, 042325 (2011))

Renyi-2 measures of Gaussian quantum correlations

(G. Adesso *et al.*, Phys. Rev. Lett. **109**, 190502 (2012))

We took a different route ...

Secret key agreement



- A, B, E obey $P(A, B, E)$
- Alice and Bob want a secret key
- They can use local operations and public communication (LOPC) PC - Eve hears messages but cannot tamper them

(U. Maurer, IEEE Trans. Inf. Theory 39, 733 (1993))

Intrinsic information

$$I\left(A:B\downarrow E\right):=\inf_{E\to\tilde{E}}\left[I\left(A:B|E\right)\right]$$

$I\left(A;B|E\right)$ - conditional mutual information; minimum over $P\left(\tilde{E}|E\right)$

(U. Maurer and S. Wolf, IEEE Trans. Inf. Theory **45**, 499 (1999))

$$S\left(A;B\|E\right)\leq I\left(A:B\downarrow E\right)\leq I_{\text{form}}\left(A;B|E\right)$$

$S\left(A;B\|E\right)$-secret key rate; $I_{\text{form}}\left(A;B|E\right)$-information of formation

$I\left(A:B\downarrow E\right)>0\Rightarrow$ no LOPC prepartion of P-secret correlations

Entanglement can be mapped onto secret correlations!

$$\rho_{AB} \to |\Psi\rangle_{ABE} \to P(A,B,E) = \mathrm{Tr}(\Pi_A \otimes \Pi_B \otimes \Pi_E |\Psi\rangle_{ABE}\langle\Psi|)$$

ρ_{AB} is entangled $\Leftrightarrow I(A:B\downarrow E) > 0$

P can inherit more properties of ρ_{AB}!

Bound entanglement $\to$ bound information

(A. Acín *et al.*, Phys. Rev. Lett. **92**, 107903 (2004))

Can we preserve also quantitative properties of states?

Classical measure of entanglement

$$E_{\downarrow}(\rho_{AB}) := \sup_{\Pi_A, \Pi_B} \inf_{\Pi_E, |\Psi\rangle} \left[I\left(A; B \downarrow E\right) \right]$$

- Vanishes on separable states
- Equal to von Neumann entropy on pure states
- Computed for Werner state (hard otherwise)

(N. Gisin and S. Wolf, Proceedings of CRYPTO 2000, 482 (2000))

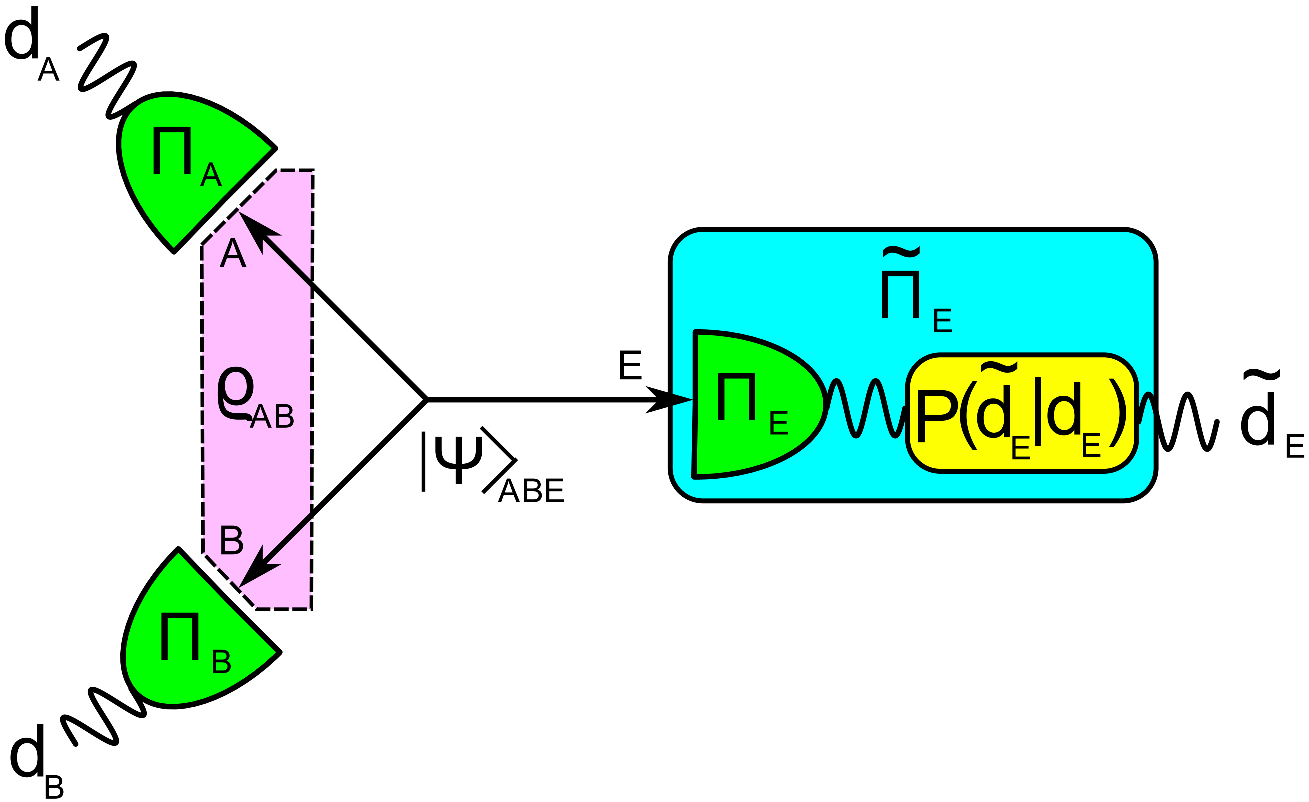
Gaussian intrinsic entanglement (GIE)

ρ_{AB} – $(N+M)$-mode Gaussian state with CM γ_{AB}

$|\Psi\rangle_{ABE}$ – Gaussian purification with CM $\gamma_\pi = \begin{pmatrix} \gamma_{AB} & \gamma_{ABE} \\ \gamma_{ABE}^T & \gamma_E \end{pmatrix}$

$\Pi_{A,B,E}$ – Gaussian measurements with outcomes d_A, d_B, d_E
and CMs $\Gamma_{A,B,E}$

$E \to \tilde{E}$ – Gaussian channel with Gaussian distribution $P(\tilde{d}_E | d_E)$



d_A
Π_A
A
ρ_AB
B
Π_B
d_B
|Ψ⟩_ABE
E
Π̃_E
Π_E
P(d̃_E|d_E)
d̃_E

$E_\downarrow$ simplifies in Gaussian scenario

$P(d_A, d_B, d_E)$ – Gaussian with CM $\quad \gamma_\pi + \Gamma_A \oplus \Gamma_B \oplus \Gamma_E$

$$I(A:B|E) \; = \; \langle I(A:B|E=e)\rangle = I(A:B|E=e) = I_{\mathrm{cond}}(A:B)$$

Mutual information of $P(d_A, d_B|d_E)$ with CM

$$\sigma_{AB} = \Gamma_A \oplus \Gamma_B + \gamma_{AB} - \gamma_{ABE}\frac{1}{\gamma_E + \Gamma_E}\gamma_{ABE}^T$$

$$\Downarrow$$

$$\boxed{I(A:B|E) = \frac{1}{2}\ln\left(\frac{\det\sigma_A \det\sigma_B}{\det\sigma_{AB}}\right)}$$

(I. M. Gelfand and A. M. Yaglom, Usp. Mat. Nauk **12**, 3 (1957))

$E \to \tilde{E}$ described by $P(\tilde{d}_E | d_E) \propto e^{-(\tilde{d}_E - X d_E)^T Y^{-1} (\tilde{d}_E - X d_E)}$,

$$\begin{aligned}\tilde{\sigma}_{AB} &= \Gamma_A \oplus \Gamma_B + \gamma_{AB} - \gamma_{ABE} X^T \left[X(\gamma_E + \Gamma_E) X^T + Y \right]^{-1} X \gamma_{ABE}^T \\ &= \text{SVD, blockwise inversion, some algebra} \\ &= \Gamma_A \oplus \Gamma_B + \gamma_{AB} - \gamma_{ABE}(\gamma_E + \tilde{\Gamma}_E)^{-1} \gamma_{ABE}^T, \tilde{\Gamma}_E - \text{CM.}\end{aligned}$$

$$\Downarrow$$

$E \to \tilde{E}$ can be integrated into Γ_E

Purifications $|\bar{\Psi}\rangle$ (K modes E) and $|\Psi\rangle$ ($R \leq K$ modes E),

$$|\bar{\Psi}\rangle_{ABE} = U_E |\Psi\rangle_{ABE} |\{0\}\rangle_{E_{R+1} \ldots E_K},$$

(L. Magnin *et al.*, Phys. Rev. A **81**, 010302 (2010))

$$\bar{\sigma}_{AB} = \ldots \bar{\gamma}_{ABE}(\bar{\gamma}_E + \bar{\Gamma}_E)^{-1} \bar{\gamma}_{ABE}^T = \ldots \gamma_{ABE}(\gamma_E + \Gamma_E)^{-1} \gamma_{ABE}^T = \sigma_{AB}$$

$$\Downarrow$$

For any $|\bar{\Psi}\rangle$ and $\bar{\Gamma}_E$ there is Γ_E on fixed $|\Psi\rangle$ giving $\bar{\sigma}_{AB} = \sigma_{AB}$

Gaussian intrinsic entanglement (GIE)

$$\boxed{E_{\downarrow}^{G}(\rho_{AB}) = \sup_{\Gamma_A,\Gamma_B} \inf_{\Gamma_E} \left[\frac{1}{2} \ln\left(\frac{\det\sigma_A \det\sigma_B}{\det\sigma_{AB}}\right)\right]}$$

$$\sigma_{AB} = \Gamma_A \oplus \Gamma_B + \gamma_{AB} - \gamma_{ABE}\frac{1}{\gamma_E + \Gamma_E}\gamma_{ABE}^{T},$$

γ_π can be CM of an arbitrary, e.g. minimal, purification.

GIE is faithful

Gaussian separable state ρ_{AB}^{sep}:

$$\rho_{AB}^{\text{sep}} = \int P_{\text{Gauss}}(\mathbf{r}) D(\mathcal{V}\mathbf{r}) |\chi_A\rangle_A\langle\chi_A| \otimes |\chi_B\rangle_B\langle\chi_B| D^\dagger(\mathcal{V}\mathbf{r}) d\mathbf{r},$$

(R. F. Werner and M. M. Wolf, Phys. Rev. Lett. **86**, 3658 (2001))

Purification:

$$|\tilde{\Psi}\rangle_{ABE} = \int \sqrt{P_{\text{Gauss}}(\mathbf{r})} D(\mathcal{V}\mathbf{r}) |\chi_A\rangle_A |\chi_B\rangle_B |\mathbf{r}\rangle_E d\mathbf{r}$$

($|\mathbf{r}\rangle_E$ – product of position eigenvectors).

Measurement of $|\mathbf{r}'\rangle_E \rightarrow$ factorized state:

$$D(\mathcal{V}\mathbf{r}')|\chi_A\rangle_A|\chi_B\rangle_B \Rightarrow \sigma_{AB} = \sigma_A \oplus \sigma_B \Rightarrow E_{\downarrow}^{G}\left(\rho_{AB}^{\text{sep}}\right) = 0.$$

One can also shown that $E_{\downarrow}^{G}(\rho_{AB}) = 0 \Rightarrow \rho_{AB}$ is separable.

GIE vanishes if and only if ρ_{AB} is separable

GIE for pure states

$$\rho_{\text{pure}} = |\psi\rangle_{AB}\langle\psi| \rightarrow |\Psi\rangle_{ABE} = |\psi\rangle_{AB}|\varphi\rangle_{E} \Rightarrow \gamma_{ABE} = 0$$

$$\Rightarrow \sigma_{AB} = \gamma_{AB} + \Gamma_A \oplus \Gamma_B,$$

$\frac{\text{det}\sigma_A \text{det}\sigma_B}{\text{det}\sigma_{AB}}$ maximized by double homodyning on A and B

(L. Mišta *et al.*, Phys. Rev. A **83**, 042325 (2011)).

$$\boxed{E_{\downarrow}^{G}(\rho_{\text{pure}}) = \ln\left(\sqrt{\text{det}\gamma_A}\right)}$$

γ_A – local CM of subsystem A.

GIE is not equal to local von Neumann entropy on pure states

Monotonicity of GIE

Gaussian local trace-preserving operations and classical communication (GLTPOCC) $\mathcal{M}$: $\rho_{AB} \to \rho_{AB}^{\mathcal{M}}$

$$E_{\downarrow}^{G}(\rho_{AB}) = \sup_{\Gamma_A, \Gamma_B} \inf_{\Gamma_E} f(\gamma_\pi, \Gamma_A, \Gamma_B, \Gamma_E),$$

$$f(\gamma_\pi, \Gamma_A, \Gamma_B, \Gamma_E) := \frac{1}{2} \ln\left(\frac{\det\sigma_A \det\sigma_B}{\det\sigma_{AB}}\right).$$

$$E_{\downarrow}^{G}\left(\rho_{AB}^{\mathcal{M}}\right) = f\left(\gamma_\pi^{\mathcal{M}}, \Gamma_A^{\mathcal{M}}, \Gamma_B^{\mathcal{M}}, \Gamma_E^{\mathcal{M}}\right),$$
$$E_{\downarrow}^{G}(\rho_{AB}) = f\left(\gamma_\pi, \Gamma_A^{(0)}, \Gamma_B^{(0)}, \Gamma_{E_\rho}^{(0)}\right).$$

$\mathcal{M}$ can be represented by a quantum state $M_{A_{\text{in}}B_{\text{in}}A_{\text{out}}B_{\text{out}}}$.

(A. Jamiołkowski, Rep. Math. Phys. 3, 275 (1972)).

$\mathcal{M}$ can be implemented by teleportation via $M_{A_{\text{in}}B_{\text{in}}A_{\text{out}}B_{\text{out}}}$.

(J. Fiurášek, Phys. Rev. Lett. 89, 137904 (2002)).

$$\rho_{AB}^{\mathcal{M}} \to |\Psi^{\mathcal{M}}\rangle \propto {}_{AA_{\text{in}}}\langle\{0\}|_{BB_{\text{in}}}\langle\{0\}|\Psi\rangle_{ABE_\rho}|M\rangle_{A_{\text{in}}B_{\text{in}}A_{\text{out}}B_{\text{out}}E_M}$$

($|M\rangle$ purifies M, $|\Psi\rangle$ purifies ρ_{AB})

If $\mathcal{M}$ is LOCC M is separable, i.e,.

$$M_{A_{\text{in}}B_{\text{in}}A_{\text{out}}B_{\text{out}}} = \sum_i p_i M^{(i)}_{A_{\text{in}}A_{\text{out}}} \otimes M^{(i)}_{B_{\text{in}}B_{\text{out}}}$$

(G. Giedke and J. I. Cirac, Phys. Rev. A 66, 032316 (2002)).

$\exists$ measurement (CM $\tilde{\Gamma}^{\mathcal{M}}_{E_M}$) projecting $|M\rangle$ to $M^{(i)}_{A_{\text{in}}A_{\text{out}}} \otimes M^{(i)}_{B_{\text{in}}B_{\text{out}}}$.

$\Rightarrow$ Independent teleportations of A and B followed by measurements $\rightarrow$ new Gaussian measurements on ρ_{AB} with CMs $\Gamma'_{A,B}$ (only if $M^{(i)}_{A_{\text{in}}A_{\text{out}}}, M^{(i)}_{B_{\text{in}}B_{\text{out}}}$ preserve the trace).

$$\begin{aligned} E^G_{\downarrow}\left(\rho^{\mathcal{M}}_{AB}\right) &\leq f\left(\gamma^{\mathcal{M}}_{\pi}, \Gamma^{\mathcal{M}}_{A}, \Gamma^{\mathcal{M}}_{B}, \Gamma^{(0)}_{E_\rho} \oplus \tilde{\Gamma}^{\mathcal{M}}_{E_M}\right) \leq f\left(\gamma_{\pi}, \Gamma'_{A}, \Gamma'_{B}, \Gamma^{(0)}_{E_\rho}\right) \\ &\leq E^G_{\downarrow}\left(\rho_{AB}\right). \end{aligned}$$

GIE does not increase under GLTPOCC

GIE for CV GHZ state

$$\gamma_{ABC}^{GHZ} = \begin{pmatrix} \alpha & \kappa & \kappa \\ \kappa & \alpha & \kappa \\ \kappa & \kappa & \alpha \end{pmatrix} \Rightarrow \gamma_{AB}^{GHZ} = \begin{pmatrix} \alpha & \kappa \\ \kappa & \alpha \end{pmatrix}$$

$\alpha = \text{diag}\left(\frac{e^{2r}+2e^{-2r}}{3}, \frac{e^{-2r}+2e^{2r}}{3}\right)$, $\kappa = \text{diag}\left(\frac{e^{2r}-e^{-2r}}{3}, -\frac{e^{2r}-e^{-2r}}{3}\right)$,
$r \geq 0$ (squeezing parameter).

(P. van Loock and S. L. Braunstein, Phys. Rev. Lett. 84, 3482 (2000))

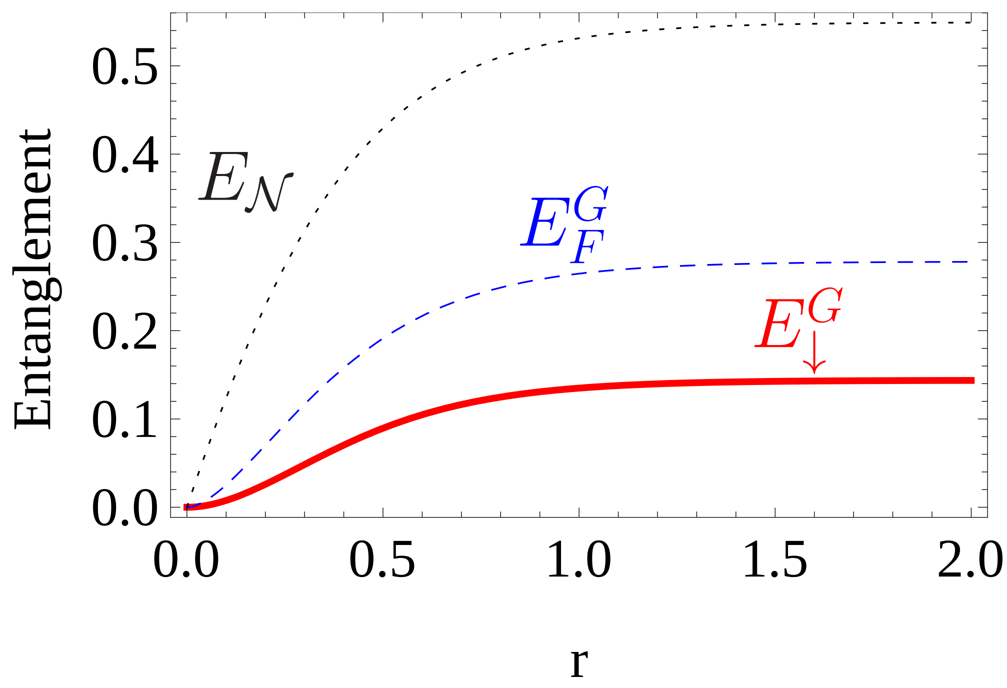
Numerics: Eve's optimal measurement is homodyning of x_E

$$\sigma_{AB} = \gamma_{AB}^{\text{cond}} + \Gamma_A \oplus \Gamma_B, \text{ where } \gamma_{AB}^{\text{cond}} \text{ is pure} \Rightarrow$$

$\frac{\text{det}\sigma_A \text{det}\sigma_B}{\text{det}\sigma_{AB}}$ maximized by double homodyning on A and B

(L. Mišta *et al.*, Phys. Rev. A 83, 042325 (2011))

$$E_{\downarrow}^{G}\left(\rho_{AB}^{GHZ}\right) = \frac{1}{2}\ln\left(\frac{2+\frac{e^{2r}}{x}+\frac{x}{e^{2r}}}{4}\right), \quad x = \frac{e^{2r}+2e^{-2r}}{3}$$



Comparison of the GIE $E_\downarrow^G$ with the Gaussian entanglement of formation E_F^G and the logarithmic negativity $E_{\mathcal{N}}$ as a function of squeezing r for the $CV\ GHZ$ state.

Conclusion

- New faithful quantifier of Gaussian entanglement.
- Operationally associated to secret key distillation.
- Monotonicity, additivity, relation to other measures etc?