Comparison of homodyne and heterodyne tomography: Optimal quantum measurements

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Outline

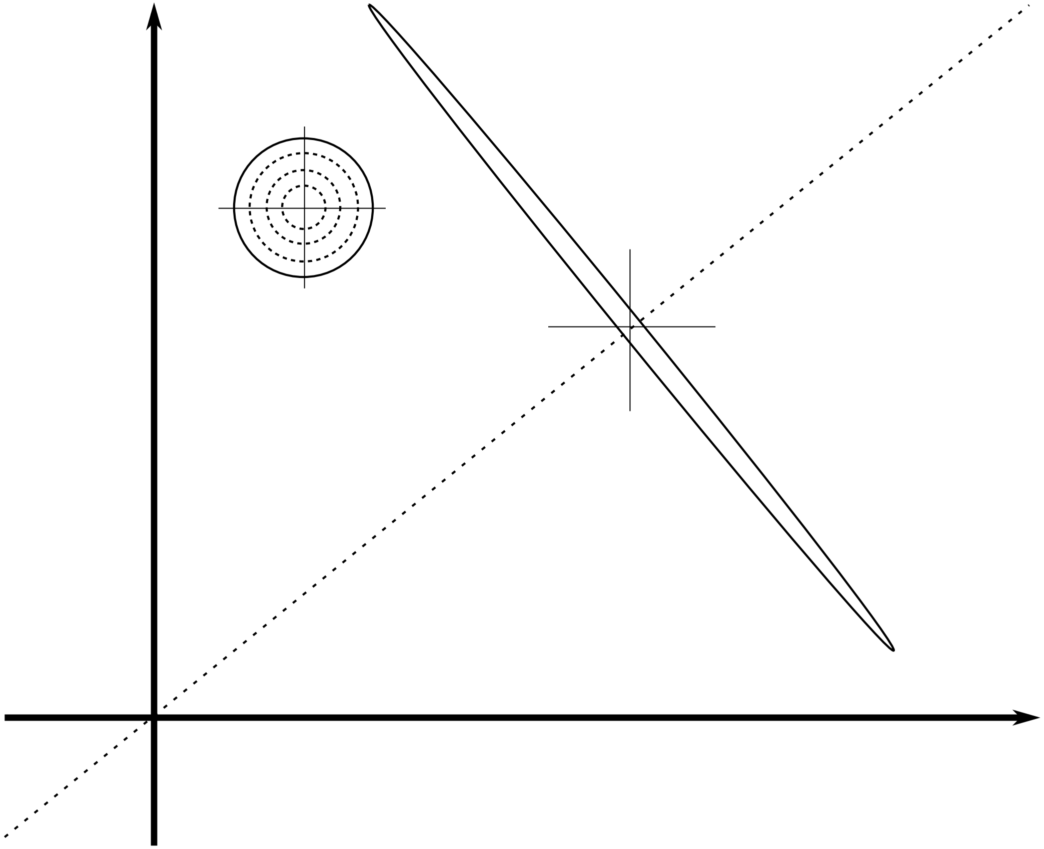
Motivation: Concept of phase space, heterodyne detection, balanced homodyne and unbalanced homodyne detection

Signal analysis: marginal vs conditional probability distributions

Tomography in phase space

Optics: From Image Processing to Wave Front Sensing

Phase space

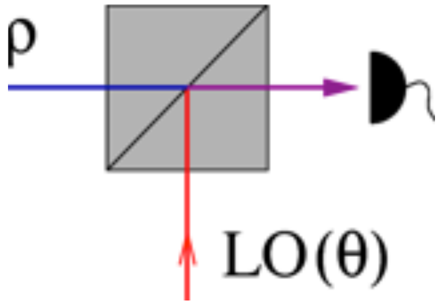


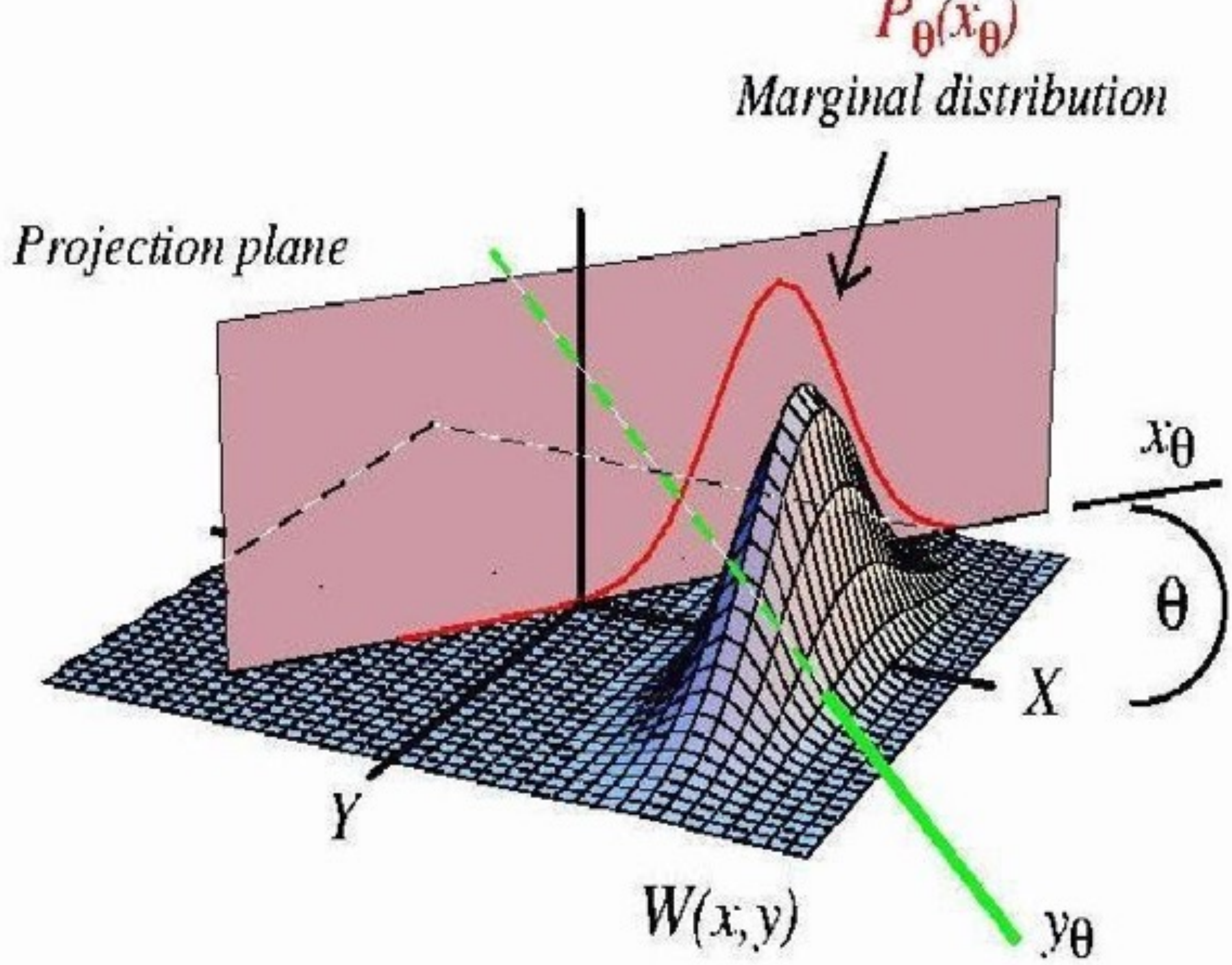
Homodyne detection

Rotated quadrature operators

$$x_\theta = \frac{1}{2}\left[a e^{-i\theta} + a^\dagger e^{i\theta}\right] \quad p_\theta = \frac{1}{2i}\left[a e^{-i\theta} - a^\dagger e^{i\theta}\right]$$

Homodyne measurement

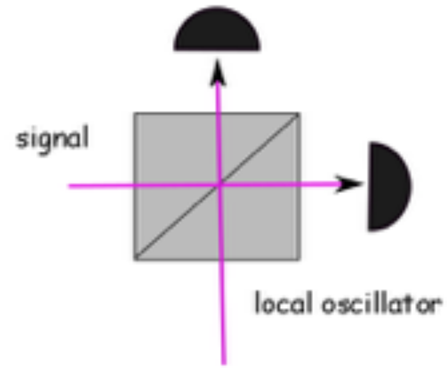




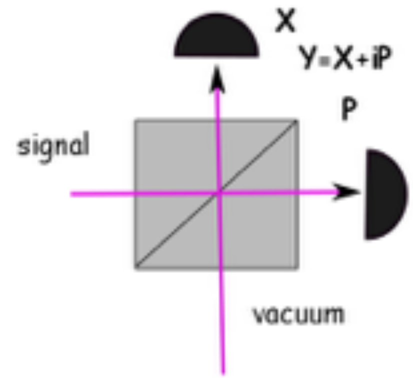
$P_\theta(x_\theta)$
Marginal distribution
Projection plane
x_θ
θ
X
Y
W(x,y)
y_θ

Heterodyne detection

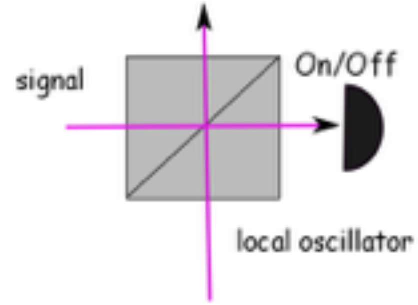
Frequency mismatch between signal and local oscillator



Double homodyne detection



Unbalanced homodyning: low transmittivity for LO



Heterodyne detection

Detection of complex amplitude

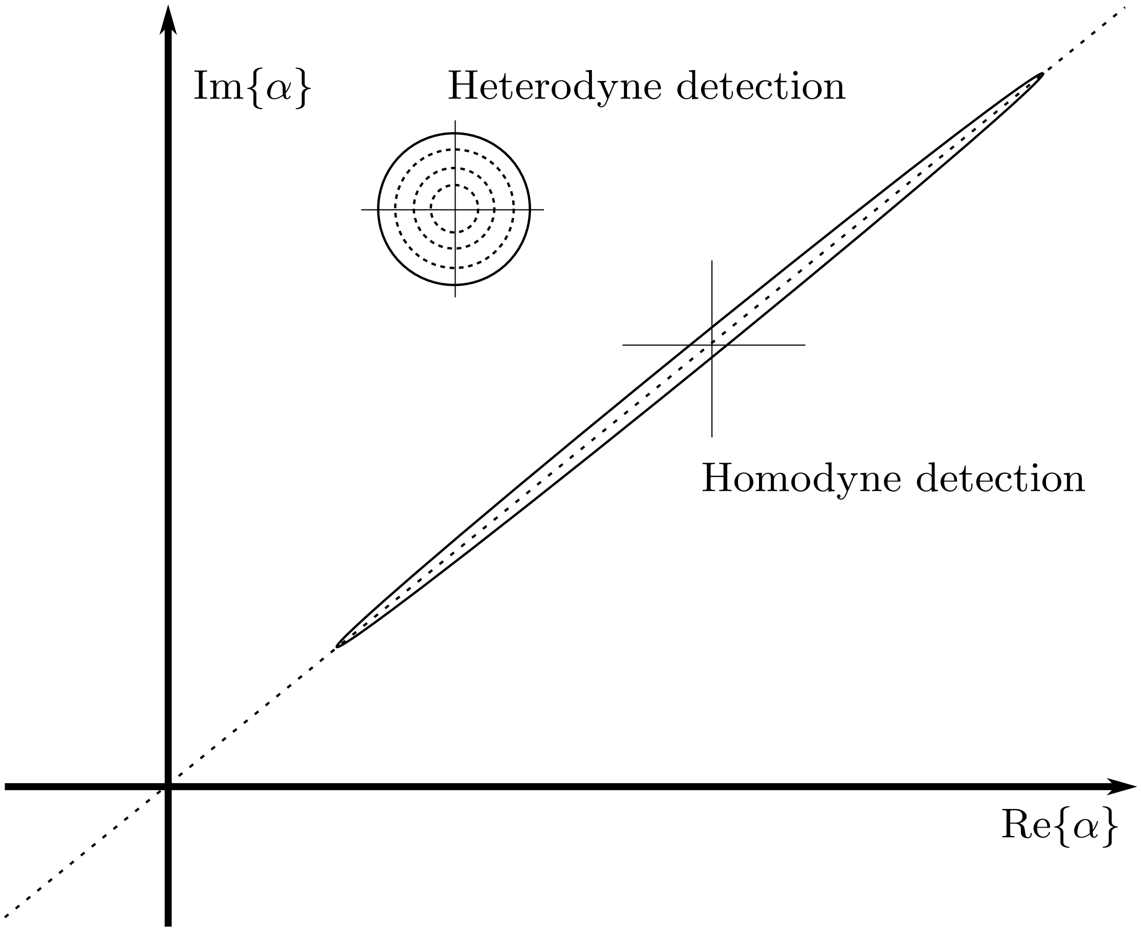
$$\mathbf{Y} = \mathbf{a}_{sig} + \mathbf{b}^{\dagger}_{aux}, \ [\mathbf{Y}, \mathbf{Y}^{\dagger}] = \mathbf{0}$$

Direct sampling of the Q-function

$$\frac{1}{\pi} \int |\alpha\rangle\langle\alpha| d^2\alpha = 1$$

Why to do heterodyne (=double homodyne) if it is more involved and more noisy than homodyne detection ?!?

Phase space



$\text{Im}\{\alpha\}$
Heterodyne detection
Homodyne detection
$\text{Re}\{\alpha\}$

Heterodyne vs heterodyne detection

Covariance matrices for Gaussian states:

$$\mathbf{G}_{\mathrm{Q}} = \mathbf{G}_{\mathrm{W}} + 1/2$$

Variance of homodyne detection: marginal distribution of Wigner function

$$\sigma_\theta^2 = u_\theta^T \mathbf{G}_{\mathrm{W}} u_\theta + \delta_\eta^2$$

Conditional variance of Q function (along the line in the phase space)

$$\Sigma_\theta^2 = \left(u_\theta^T [\mathbf{G}_{\mathrm{Q}} + \delta_\eta^2]^{-1} u_\theta \right)^{-1}$$

Noise term:

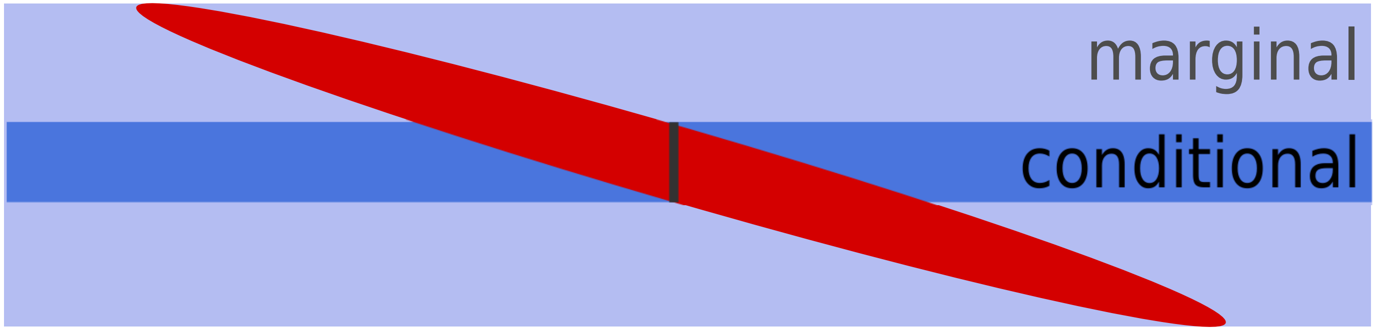
$$\delta_\eta^2 = (1-\eta)/2\eta$$

Geometrical relation between marginal and conditional distributions

Relation for generic covariance matrix G and orthogonal basis vectors u, v

$$\sigma_\theta^2 = u_\theta^T \mathbf{G} u_\theta \equiv G_{uu,\theta}$$

$$\Sigma_\theta^2 = \left(u_\theta^T \mathbf{G}^{-1} u_\theta\right)^{-1} = \frac{G_{uu,\theta} G_{vv,\theta} - G_{uv,\theta}^2}{G_{vv,\theta}} \leq G_{uu,\theta} = \sigma_\theta^2 \, .$$



Noise analysis for minimum uncertainty states

Wigner covariance matrix

$$\mathbf{G}_{\mathrm{W}} \widehat{=} \begin{pmatrix} \frac{1}{2\lambda} & 0 \\ 0 & \frac{\lambda}{2} \end{pmatrix}$$

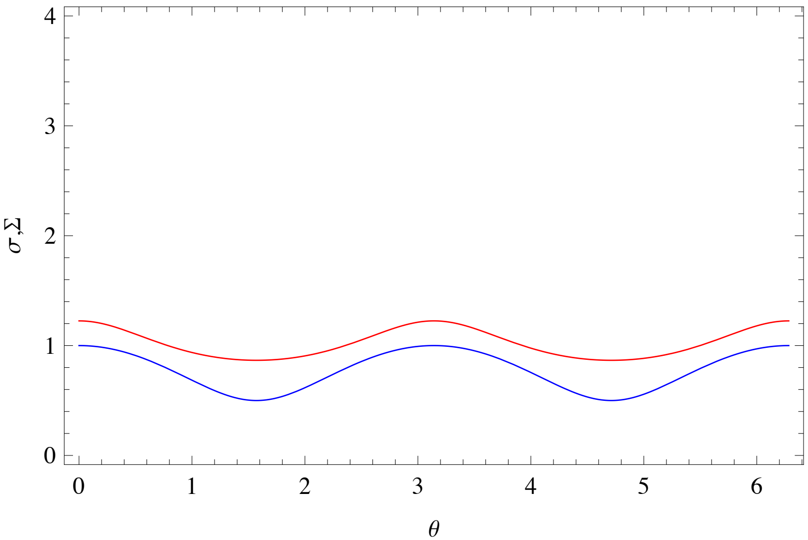
Marginal variance

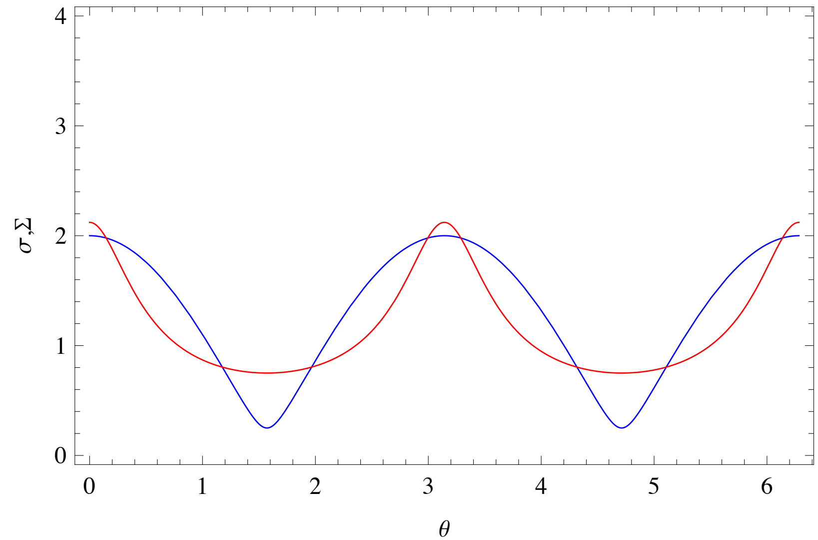
$$\sigma_\theta^2 = \frac{1}{2\lambda}(\cos\theta)^2 + \frac{\lambda}{2}(\sin\theta)^2$$

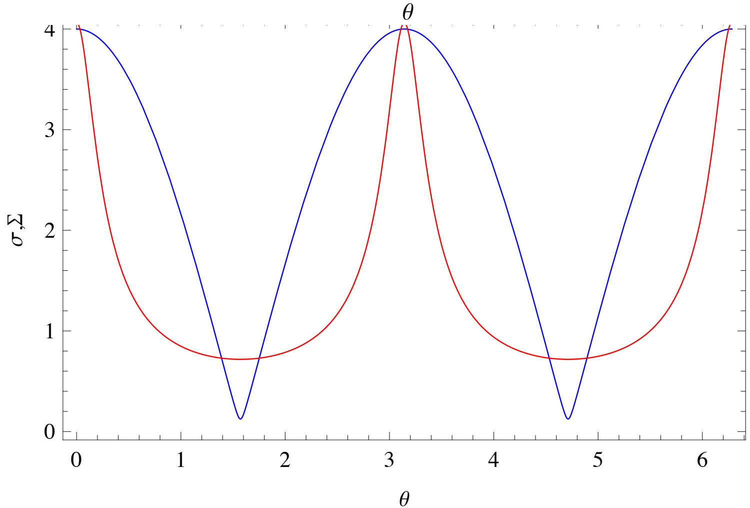
Conditional variance

$$\Sigma_\theta^2 = \left[1 + \frac{\lambda - 1}{\lambda + 1}\cos(2\theta)\right]^{-1}$$

Noise comparison







BLUE: marginal
for homodyne detection
RED: conditional
for heterodyne detection

Rationale behind the noise analysis

Without vacuum term conditional variance would be always better than marginal distribution

Heterodyne detection (=simultaneous detection) allows to define variable which could be sometimes less noisy than corresponding homodyne detection! See in experiment !!!

Noise analysis by itself does not say anything about the overall performance of homodyne vs heterodyne detections. Why?

Tomography is needed for meaningful comparison! = You should know what the detection means…

Tomography in phase space

Homodyne detection: Samples the variances of Wigner function, the covariance must be reconstructed (more involved reconstruction)

Heterodyne detection: The Q function is sampled directly (direct reconstruction)

Tomography = Detected noise + error from inversion

Mathematical tools for reconstruction of covariance matrix

Estimated covariance

$$\mathbf{G}_{\mathrm{W}} \widehat{=} \begin{pmatrix} g_1 & g_3/\sqrt{2} \\ g_3/\sqrt{2} & g_2 \end{pmatrix}$$

Hilbert-Schmidt distance

$$H = \overline{\left(\mathbf{G}_{\mathrm{W}} - \widehat{\mathbf{G}}_{\mathrm{W}}\right)^2} = \sum_k \overline{(g_k - \widehat{g}_k)^2}.$$

Cramer- Rao bound

$$H \geq Sp\mathbf{F}^{-1}$$

Fisher information

$$\mathbf{F} = \frac{N}{2} Sp\, \mathbf{G}^{-1} \frac{\partial \mathbf{G}}{\partial \mathbf{g}} \mathbf{G}^{-1} \frac{\partial \mathbf{G}}{\partial \mathbf{g}}$$

Rationale behind the noise analysis

Performance of homodyne tomography

$$H_{\rm hom} = \frac{2\,Sp\ \mathbf{G}_{\rm hom}\left(Sp\ \mathbf{G}_{\rm hom} + 3\sqrt{Det\ \mathbf{G}_{\rm hom}}\right)}{N}$$

Performance of heterodyne detection

$$H_{\rm het} = \frac{2\big[(Sp\ \mathbf{G}_{\rm het})^2 - Det\ \mathbf{G}_{\rm het}\big]}{N}$$

Measurement in Phase Space

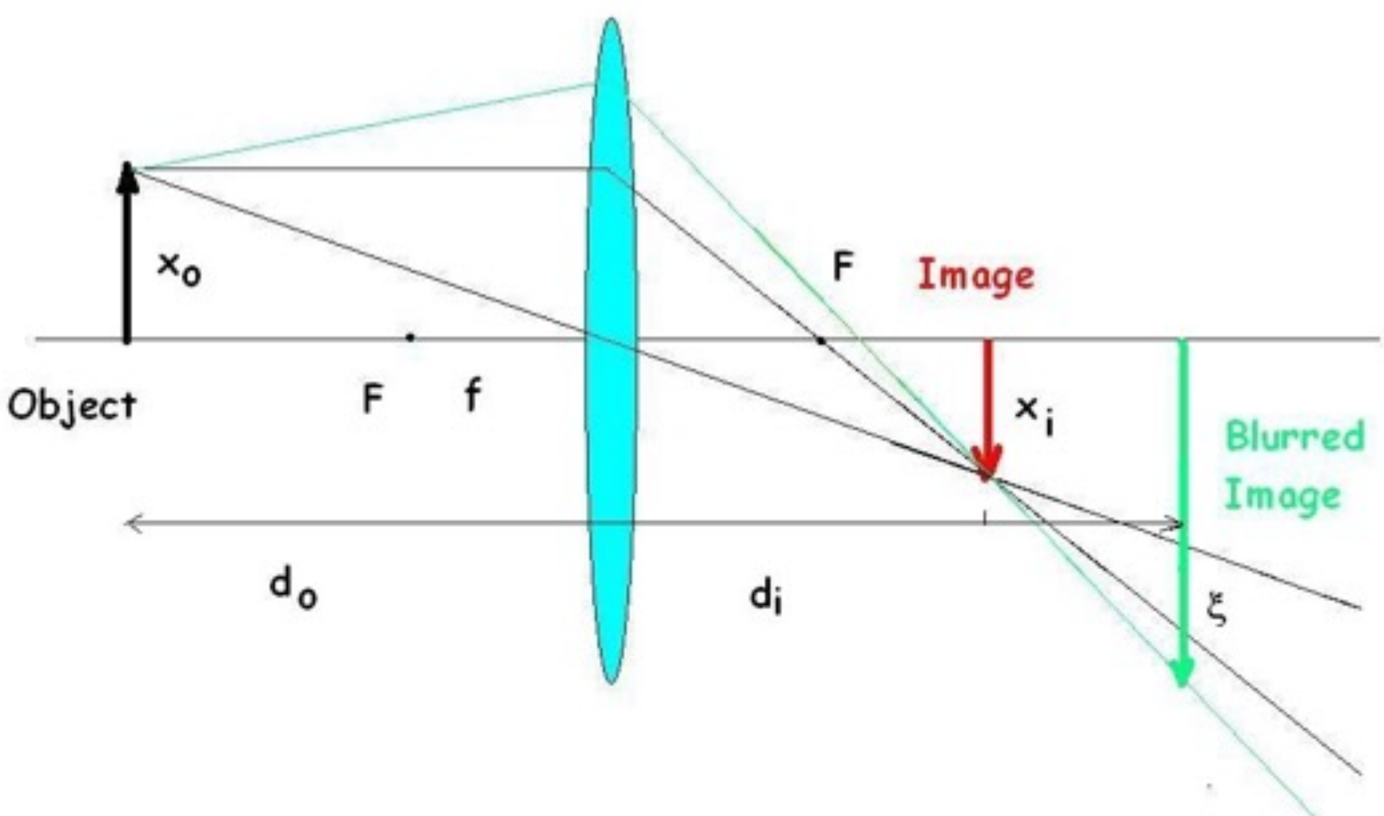
Homodyne detection - Projection into the Rotated Quadrature Eigenstates

Heterodyne detection - Projection into the coherent state basis with fluctuating position in phase space

Unbalanced homodyning - Projection into the coherent state basis with prefixed position in phase space

Phase space in optics

Optical Imaging: Lens equation in geometrical optics



Lens equation in geometrical optics:

$$1/d_o + 1/d_i = 1/f$$

For sharp image: $x_i = M\, x_o$,

magnification $M = d_i/d_o$

For the blurred image: $\xi = \alpha\, x_o + \beta\, p_o$

x_o … position of the ray

$p_o = 2\pi/\lambda\ \theta$ … direction of the ray

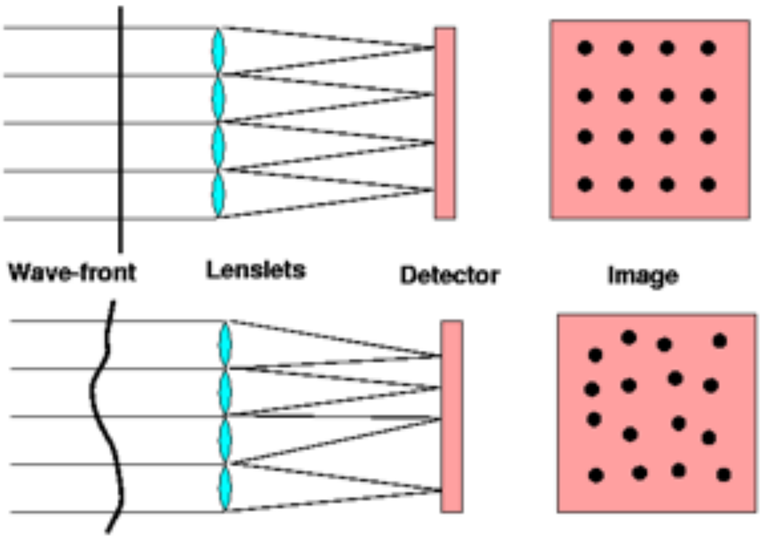
Meaning in quantum mechanics:

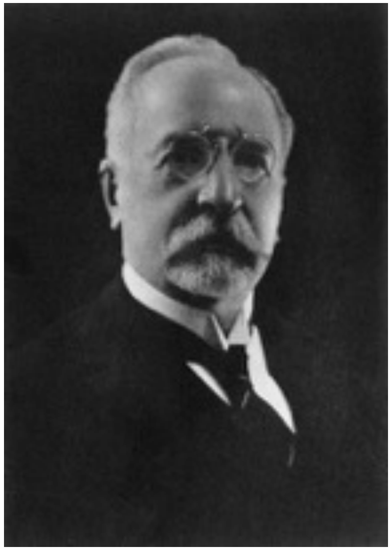
Rotated quadrature operator for $[x_o, p_o] = i\hbar$

See the analogy with the free evolution

$$x(t) = x(0) + p(0)\ t/m$$

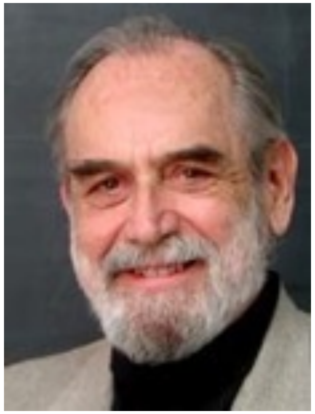
Scanning of the optical field: Hartmann-Shack sensor





Johannes Hartmann (1865-1936)

Roland Shack (1970's)



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Wavefront sensing reveals optical coherence

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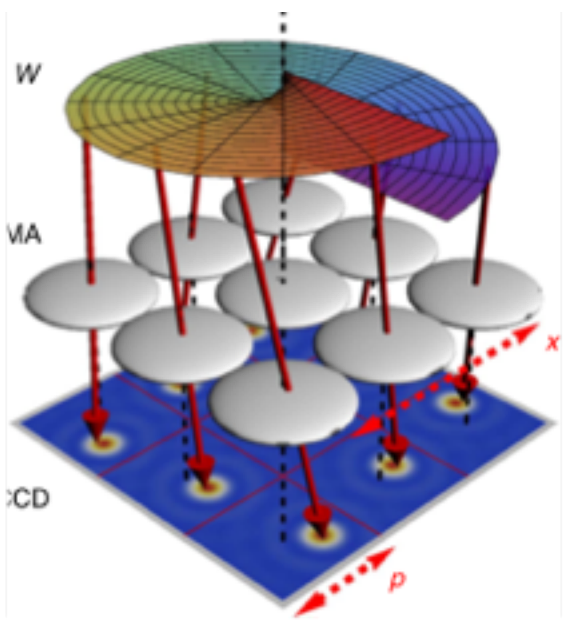


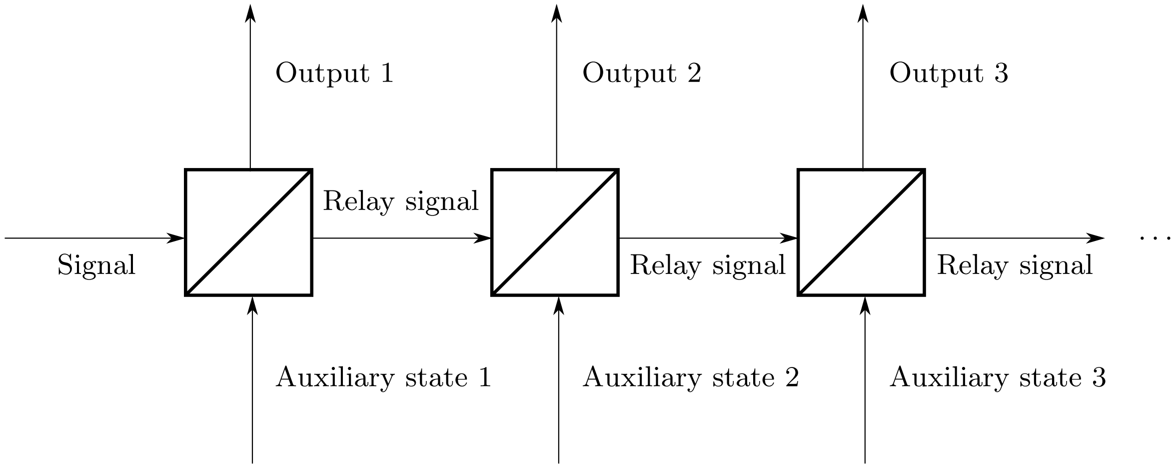
Conclusions

Loosely speaking performance heterodyne tomography outperforms the homodyne one if the vacuum noise is unimportant !

The improvement is NOT (only?) due to the noise reduction !!

It is worth to investigate the heterodyne detection experimentally !!!

It is worth to investigate the heterodyne detection experimentally



Thanks for your attention!