Nové poznatky o konceptuálních svazech s neúplnou informací a o neúplné informaci vůbec

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Introduction

Genesis

- Concept lattices with incomplete information

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Our approach

- Mathematical structures with incomplete information
- Incomplete information: equality of elements, membership in sets
- Treating all possible worlds at once, as single abstract (incomplete) world
- Ignorance embodied into structure of truth values

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Influences

- Fuzzy sets
- Boolean valued models
- Possible worlds

Outline

Boolean algebra of conditions (“structure of ignorance”)

Conditions

- Information is of binary nature
- Missing bits of information determined by external conditions
- Complete Boolean algebra

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Realities

- Completion of unknown information: complete homomorphism $h\colon L \to K$
- Determines (partially) a possible world, reality
- $h(c) = 1$: c *is satisfied in* h
- $h\colon L \to \mathbf{2}$: *total reality*

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Boolean algebra of conditions L

- Complete atomic Boolean algebra
- Construction: Lindenbaum algebra, admissible evaluations

Conditional universes (“ignorance of equality”)

- Conditional universe: the underlying set of a structure
- with (incomplete) information on equality of elements
- $x_1 \approx x_2 \in L$: condition for “$x_1 = x_2$”

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Definition

An *L-conditional universe* is a set X together with an L-equality $\approx$, i.e. a mapping $\approx: X \times X \to K$ satisfying

$$x \approx x = 1, \quad \text{(reflexivity)}$$
$$x_1 \approx x_2 = x_2 \approx x_1, \quad \text{(symmetry)}$$
$$(x_1 \approx x_2) \wedge (x_2 \approx x_3) \leq x_1 \approx x_3, \quad \text{(transitivity)}$$
$$x_1 \approx x_2 = 1 \quad \text{implies} \quad x_1 = x_2. \quad \text{(separation)}$$

- *Subuniverses*, *products*

Realizations of universes

- For a total reality h, X should transform to an ordinary set with ordinary equality
- X becomes X^h, $x \in X$ becomes $x^h \in X^h$ ... realization
- Equal elements of X should "glue" together
- Equal elements: $h(x_1 \approx x_2) = 1$

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$$h(x_1 \approx x_2) = x_1^h \approx^h x_2^h \qquad (*)$$

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Let $h\colon L \to K$ be a reality. An *h-realization of* $\langle X, \approx \rangle$ is a K-conditional universe $\langle X^h, \approx^h \rangle$ together with a surjective mapping $X \to X^h$, $x \mapsto x^h$ satisfying $(*)$.

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- $(*)$: "$x_1 = x_2$ in h iff *it is satisfied in h that* $x_1 = x_2$"
- Moreover: if "$x_1 = x_2$ in each total h" then $x_1 = x_2$
- All h-realizations are isomorphic
- X^h may be obtained by factorization

Conditional sets

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Definition

Conditional set A *in* X is an L-set $A\colon X \to L$.

- $A(x)$: *membership condition*
- Conditional relations

Realizations of conditional sets

- We are going to define for a conditional set A in X its realization A^h in X^h
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Proposition

If A is extensional then $(*)$ holds true.

- Moreover...

Power relations

- Lifting binary relations on X to L^X

Definition

Let R be a binary conditional relation on X. For conditional sets A, B in X we set

$$R^{\rightarrow}(A,B) = \bigwedge_{x_1 \in X} \Big(A(x_1) \rightarrow \bigvee_{x_2 \in X} R(x_1, x_2) \wedge B(x_2) \Big),$$

$$R^{\leftarrow}(A,B) = \bigwedge_{x_2 \in X} \Big(B(x_2) \rightarrow \bigvee_{x_1 \in X} A(x_1) \wedge R(x_1, x_2) \Big)$$

$$R^{+}(A,B) = R^{\rightarrow}(A,B) \wedge R^{\leftarrow}(A,B).$$

Extensional equality

- Consider the lifted relation $\approx^+$
- $\approx^+$ is reflexive, symmetric and transitive, separated on extensional sets
- $A^h \approx^{h+} B^h = h(A \approx^+ B)$ even for non-extensional A, B

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Theorem (extensional equality)

The following three conditions are equivalent for any two conditional sets A, B in X.

1. $A \approx^+ B = 1$,
2. $\mathrm{C}_{\approx} A = \mathrm{C}_{\approx} B$,
3. A *equals* B *in any total reality.*

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Definition

A, B above are called *extensionally equal*.

- Challenge: find a minimal crisp subset $Y \subset X$ extensionally equal with X (and, possibly, satisfying additional conditions).
- Y will not be extensional.

Conditional bijection

- Generalization of extensional equality
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Definition

Conditional set $\mathbf{x}$ in X is called a *conditional point* if it is a block of $\approx$:

$$\mathbf{x}(x_1) \wedge \mathbf{x}(x_2) \leq x_1 \approx x_2.$$

$\mathbf{x}$ is *proper* if its *height* $\bigvee_{x \in X} \mathbf{x}(x)$ is 1.

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Definition

Conditional relation F between X and Y is a *conditional mapping* if for each proper conditional point $\mathbf{x}$ in X, $F(\mathbf{x})$ is a proper conditional point.
F is a *conditional bijection* if, in addition, F^{-1} is a conditional mapping.

Conditional bijection (remarks)

Conditional points

- For total h, $\mathbf{x}^h$ has at most 1 element
- If $\mathbf{x}^h = \emptyset$, we say that $\mathbf{x}$ does not exist in h
- $\mathbf{x}$ is proper iff it exists in each total reality

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Conditional mappings

- F is a conditional mapping iff it is a (ordinary) mapping in each total reality
- F is a conditional bijection iff it is a (ordinary) bijection in each total reality
- Crisp subsets $Y_1, Y_2 \subseteq X$ are extensionally equal iff $\approx \cap (Y_1 \times Y_2)$ is a conditional bijection $Y_1 \to Y_2$

Outline

Conditional ordered sets

- Conditional order: example of a conditional structure

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Definition

A *conditional order* on an L-conditional universe $\langle U, \approx \rangle$ is an extensional binary conditional relation $\preceq$ which is reflexive and transitive and satisfies

$$(u_1 \preceq u_2) \wedge (u_2 \preceq u_1) \leq u_1 \approx u_2. \qquad \text{(antisymmetry)}$$

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- For extensional $\preceq$ also vice versa

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- $\preceq$ is a (partial) order in each total reality
- For extensional $\preceq$ also vice versa
- $\preceq^h$ is unique; extensionality gives

$$u_1^h \preceq^h u_2^h = h(u_1 \preceq u_2)$$
$$V_1^h \preceq^{h+} V_2^h = h(V_1 \preceq^+ V_2)$$

V_1, V_2 need not be extensional

Isotone conditional mappings

Definition

A conditional mapping $F\colon U \to V$ is *isotone* if for each $u_1, u_2 \in U$

$$(u_1 \preceq_U u_2) \wedge F(u_1, v_1) \wedge F(u_2, v_2) \leq v_1 \preceq_V v_2. \qquad \text{(isotony)}$$

F is a *conditional isomorphism* if F^{-1} is isotone as well.

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Proposition

The following three conditions are equivalent.

1. F *is isotone.*
2. $\mathbf{u}_1 \preceq^+ \mathbf{u}_2 \leq F(\mathbf{u}_1) \preceq^+ F(\mathbf{u}_2)$ *for any two proper conditional points* $\mathbf{u}_1, \mathbf{u}_2 \subseteq U$.
3. F *is an isotone mapping of ordered sets in each total reality.*

Suprema and infima

Upper and lower cones

$$\mathcal{U}V(v) = \bigwedge_{u \in U} V(u) \to (u \preceq v) \qquad \mathcal{L}V(v) = \bigwedge_{u \in U} V(u) \to (v \preceq u)$$

$$(\quad \mathcal{U}V(v) = V \preceq^{\to} \{v\} \qquad \mathcal{L}V(v) = \{v\} \preceq^{\leftarrow} V \quad)$$

Supremum and infimum

$$\operatorname{Sup} V(u) = \mathcal{U}V(u) \wedge \mathcal{L}\mathcal{U}V(u), \qquad \operatorname{Inf} V(u) = \mathcal{L}V(u) \wedge \mathcal{U}\mathcal{L}V(u)$$

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- $\operatorname{Sup} V$ and $\operatorname{Inf} V$ are conditional points

Proposition

For each reality h:

$$(\operatorname{Sup} V)^h = \operatorname{Sup} V^h, \qquad (\operatorname{Inf} V)^h = \operatorname{Inf} V^h.$$

- V need not be extensional

Conditional complete lattices

Proposition

The following conditions are equivalent:

1. $\preceq$ *is a complete lattice order in each total reality.*
2. *For each conditional set* V, $\mathrm{Inf}\, V$ *is a proper conditional point.*
3. *For each conditional set* V, $\mathrm{Sup}\, V$ *is a proper conditional point.*

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Definition

If the above three conditions are satisfied, $\langle\langle U, \approx\rangle, \preceq\rangle$ is called a *conditional complete lattice*.

- $\langle\langle U, \approx\rangle, \preceq\rangle$ need not be a completely lattice L-ordered set

Outline

Conditional context, conditional concepts

- Conditional context: "incomplete context"

Definition

L-conditional formal context is a triple $\langle X, Y, I\rangle$ where X and Y are L-conditional universes with associated L-equalities $\approx_X$ and $\approx_Y$, respectively; and $I\colon X \times Y \to L$ is an extensional conditional relation.

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- $\langle X, Y, I\rangle^h = \langle X^h, Y^h, I^h\rangle$: h-realization of $\langle X, Y, I\rangle$

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- Conditional concepts: pairs that realize to concepts

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Let A and B be extensional conditional sets in X and Y, respectively. We call the pair $\langle A, B\rangle$ a *conditional concept of* $\langle X, Y, I\rangle$ if for each total reality h the pair $\langle A, B\rangle^h = \langle A^h, B^h\rangle$ is a concept of $\langle X, Y, I\rangle^h$.

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- The set of all conditional concepts is $\mathcal{B}(X, Y, I)$

Conditional concept lattices

- The definition is in accordance with our approach
- Any set of conditional concept that realizes to the respective concept lattices is a conditional concept lattice
- Thus, there are several conditional concept lattices of $\langle X, Y, I \rangle$, all extensionally equal

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Definition

By a *conditional concept lattice of the context* $\langle X, Y, I \rangle$ we understand any set U of its conditional concepts (i.e. crisp subset $U \subseteq \mathcal{B}(X, Y, I)$) which is extensionally equal to $\mathcal{B}(X, Y, I)$. $\mathcal{B}(X, Y, I)$ itself is called *the maximal conditional concept lattice of* $\langle X, Y, I \rangle$.

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- For each total reality h we have $U^h = \mathcal{B}(X^h, Y^h, I^h)$
- We can describe easily ordering of concepts in each total reality from the structure of U ...
- ... as well as suprema and infima

Basic theorem of conditional concept lattices

Basic theorem of conditional concept lattices

Theorem

1. Any conditional concept lattice U of $\langle X, Y, I\rangle$ is a conditional complete lattice. Suprema and infima in U are given by

$$\operatorname{Sup} M(\langle A, B\rangle) = B \approx_Y^+ \bigcap M_Y, \qquad \operatorname{Inf} M(\langle A, B\rangle) = A \approx_X^+ \bigcap M_X.$$

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2. A conditional complete lattice V is conditionally isomorphic with a conditional concept lattice of $\langle X, Y, I\rangle$ iff there exist conditional mappings $\gamma\colon X \to V$ and $\mu\colon Y \to V$ such that $\gamma(X)$ is Sup-dense in V, $\mu(Y)$ is Inf-dense in V and $I(x, y) = \gamma(x) \preceq^+ \mu(y)$.

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- Part 2 provides an easy way to tell the structure of each of the possible concept lattices and to reconstruct the relation I from a diagram
- γ and μ need not be extensional

Construction of conditional concept lattices

- Crisply generated concepts: infima of crisp sets are easy
- Closure to a complete sublattice of $\mathcal{B}(X, Y, I)$: both suprema and infima are easy

Next steps

- Add non-existence (non-reflexive equality)
- Theory for other structures (finding minimal universes)
- Heyting algebras? Residuated lattices?
- Describing L by formulas of a predicate logic