Synthetic Gauge Structures & Wilson Loops with Internal & External States

Kunal K. Das

Kutztown University of Pennsylvania

Outline

- *Brief primer on Gauge Fields, Berry Phase and Wilson loops*
- *A new cyclic scheme to implement with cold atoms*
- *What is really non-Abelian? Necessity for Wilson loops*
- *A way to measure the full Wilson matrix*

Brief primer on Gauge Fields, Geometric Phase and Wilson loops

$$e^A B e^{-A} = B + [A, B]$$

Gauge Transformation

- In quantum mechanics, only the expectation is physical $\langle \psi | \hat{p} | \psi \rangle$

- So if the wave function is multiplied by a phase factor $\psi \rightarrow e^{-ie\theta} \psi$ then so should the operator $\hat{p} = -i\hbar\nabla$

$$\hat{p} \rightarrow e^{-ie\theta} \hat{p} e^{ie\theta} \quad = \hat{p} - ie[\theta, \hat{p}]$$

- Reconsider the extra term as part of a new 'gauge' potential

$$(p + eA) \rightarrow e^{-ie\theta}(p + eA)e^{ie\theta} \quad = p + e(A + \hbar\nabla\theta)$$

Geometric Phase

- Expand state in instantaneous eigenstates $\Psi(t) = \sum_i w_i(t)\Phi_i(t)$
- Assume adiabatic evolution so it follows one eigenstate $\Psi(t) \simeq w\Phi$
- Assume it has zero eigenvalue, so $i\hbar d_t[w\Phi] = H(t)[w\Phi] = 0$
- Left multiply by Φ and integrate to get $d_t w = -\langle\Phi|d_t|\Phi\rangle w$
- The formal solution $w(t) = e^{i\int dt A}$
- Time dependence is not relevant, depends on path ***C*** in parameter space ***R***

$$\gamma = i\int_C dR \cdot \langle\Phi|\nabla|\Phi\rangle$$

Wilson Matrix

- Assume TWO degenerate dark states $\Psi(t) \simeq \sum_i w_i \Phi_i$
- Then, the relations from before $d_t w = iAw$ $\quad A = i\langle \Phi | d_t | \Phi \rangle$

  are replaced by **vector** versions $d_t w_i = iA_{jk} w_k$ $\quad A_{jk} = i\langle \Phi_j | \nabla | \Phi_k \rangle$

- Make a **matrix** of the amplitudes based on which dark state is the **initial** state $d_t w_{il} = iA_{jk} w_{kl}$

  …written as a matrix equation $d_t W = iAW$

- We call this the **Wilson matrix** $\boxed{W = \mathrm{P}e^{i\int dtA}}$

Synthetic Gauge Field

- If we phase-multiply the eigenstate $|\Phi\rangle \rightarrow e^{-ie\theta}|\Phi\rangle$
  we get
  $$A = i\langle\Phi|\nabla|\Phi\rangle \rightarrow i\langle\Phi|\nabla|\Phi\rangle + e\nabla\theta$$
  which is exactly the form for U(1) Abelian gauge transformation
  $$\boxed{A \rightarrow A + e\nabla\theta}$$

- For the degenerate case, an unitary transformation $|\Phi\rangle \rightarrow U|\Phi\rangle$
  leads to
  $$A = i\langle\Phi|\nabla|\Phi\rangle \rightarrow i\langle\Phi U^\dagger|\nabla|U\Phi\rangle = iU\langle\Phi|\nabla|\Phi\rangle U^\dagger + iU^\dagger\nabla U$$
  which is exactly the form for a non-Abelian gauge transformation
  $$\boxed{A \rightarrow UAU^\dagger + iU^\dagger\nabla U}$$

Gauge invariance and Wilson loop

- For an **open** curve, like any phase the geometric phase is arbitrary due to the 'gauge freedom' to choose the reference phase

- For a **closed** curve, since the state has to be single-valued, the freedom is removed, the geometric phase becomes 'gauge invariant'

$$w(t) = e^{i \oint dt A} \qquad \gamma = \oint dt A = \int\int dS \cdot \boxed{(\nabla \times A)}$$

- For the non-Abelian case, we have likewise,

$$W_\circ = \mathrm{P} e^{i \oint dt A}$$

- The **trace** of the matrix is called the **Wilson loop** $\boxed{\mathcal{W} = \mathrm{tr}[W_\circ]}$

A new cyclic scheme to implement with cold atoms

One Ring to Rule them All …

- A new cyclic Hamiltonian for synthetic gauge field

$$H = \frac{\hbar}{2}\begin{pmatrix} 0 & e^{i\varphi_1}p & 0 & e^{-i\varphi_4}p \\ e^{-i\varphi_1}p & \delta & e^{i\varphi_2}q & 0 \\ 0 & e^{-i\varphi_2}q & 0 & e^{i\varphi_3}q \\ e^{i\varphi_4}p & 0 & e^{-i\varphi_3}q & \mp\delta \end{pmatrix}$$

- The phases need to add up to 0 or multiples of 2 π
- One dark state when detunings have the same sign: $H_{44} = +\delta$

Eigenvalues: $\{0, \frac{1}{2}\delta, \frac{1}{4}(\delta \pm \sqrt{\delta^2 + 8\,(p^2+q^2)})\}$

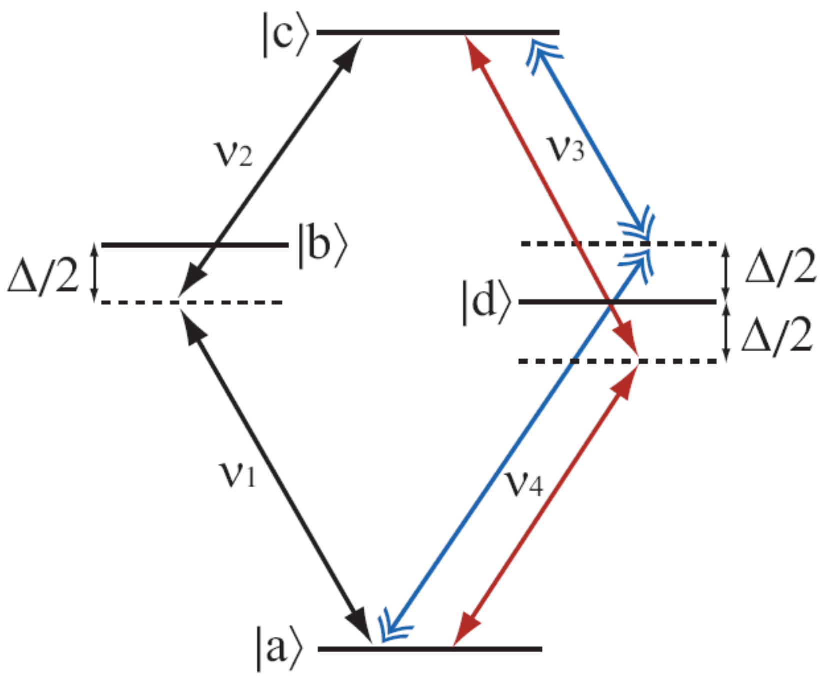
- Two dark states when detunings have opposite signs: $H_{44} = -\delta$

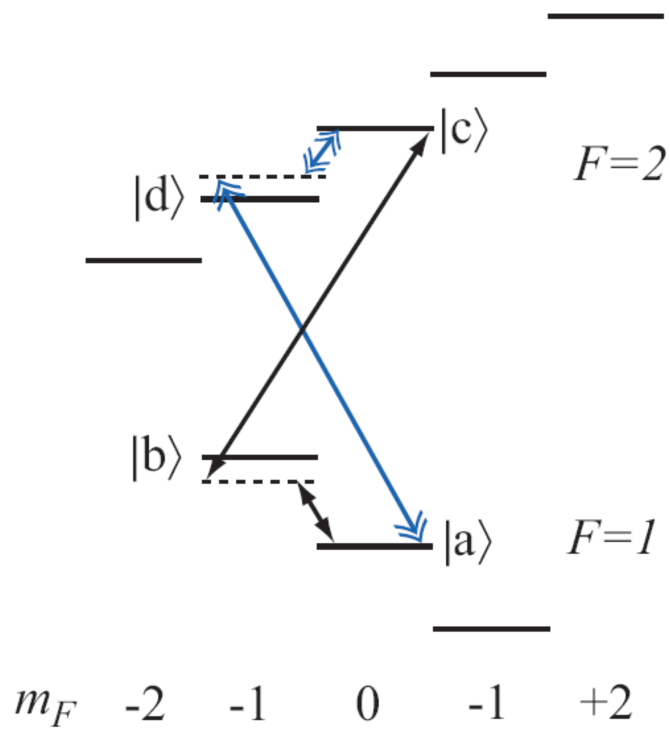
Eigenvalues : $\{0, 0, \pm\frac{1}{2}\sqrt{\delta^2 + 2\,(p^2+q^2)}\}$

Physical Implementation I

Abelian : Red
Non-Abelian: Blue

$$H_1 = \frac{\hbar}{2}\begin{bmatrix} 0 & \Omega_1^* & 0 & \Omega_4^* \\ \Omega_1 & \Delta & \Omega_2^* & 0 \\ 0 & \Omega_2 & 0 & \Omega_3 \\ \Omega_4 & 0 & \Omega_3^* & \mp\Delta \end{bmatrix}$$

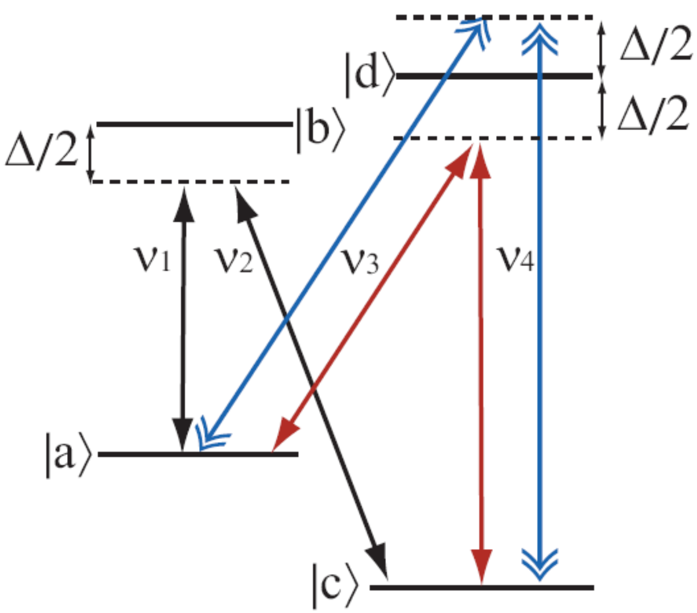


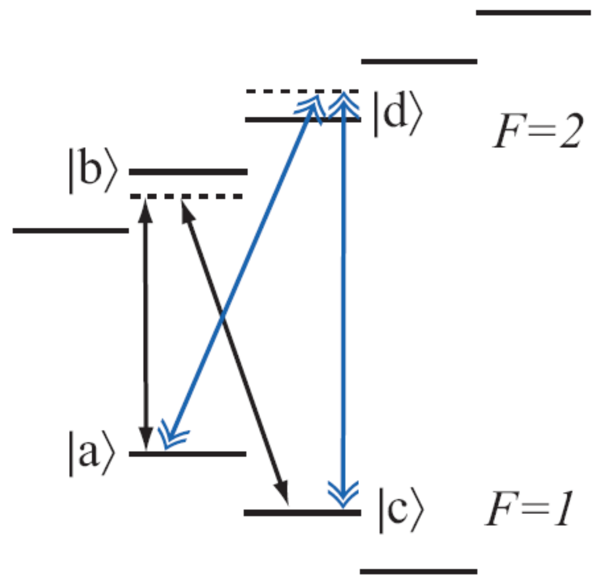


Physical Implementation II

Abelian : Red
Non-Abelian: Blue

$$H_2 = \frac{\hbar}{2}\begin{bmatrix} 0 & \Omega_1^* & 0 & \Omega_4^* \\ \Omega_1 & \Delta & \Omega_2 & 0 \\ 0 & \Omega_2^* & 0 & \Omega_3^* \\ \Omega_4 & 0 & \Omega_3 & \mp\Delta \end{bmatrix}$$





Hamiltonian and Parameter Space

- We pick a slightly restricted form of the Hamiltonian for simulation

$$H = \frac{\hbar}{2} \begin{pmatrix} 0 & e^{i\alpha}p & 0 & e^{i\alpha}p \\ e^{-i\alpha}p & \delta & q & 0 \\ 0 & q & 0 & q \\ e^{-i\alpha}p & 0 & q & \mp\delta \end{pmatrix}$$

- Reparametrize

$$\Omega = \sqrt{\delta^2 + 2(p^2+q^2)} \qquad \delta = \Omega\sin(\phi)$$
$$p = \frac{1}{\sqrt{2}}\Omega\sin(\theta)\cos(\phi) \qquad q = \frac{1}{\sqrt{2}}\Omega\cos(\theta)\cos(\phi)$$

- Note the gauge structures are NOT in physical space but in the space of parameters, $\{p,q,\alpha,\delta\}$ or equivalently $\{\theta,\phi,\alpha\}$

Dark States

- The non-Abelian degenerate dark states are coupled

$$\Phi_1^N = \left(e^{i\alpha}\cos(\theta), 0, -\sin(\theta), 0\right)$$
$$\Phi_2^N = \left(e^{i\alpha}\sin(\phi)\sin(\theta), -\frac{1}{\sqrt{2}}\cos(\phi), \cos(\theta)\sin(\phi), \frac{1}{\sqrt{2}}\cos(\phi)\right)$$

- By varying the detuning (hidden in ϕ) these states can be continuously morphed to the Abelian case, where the two states decouple and there is only one dark state

$$\Phi_1^A = \left(e^{i\alpha}\cos(\theta), 0, -\sin(\theta), 0\right)$$
$$\Phi_2^A = \left(0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),$$

Abelian Potential and Field

- In the Abelian case, in the space of the single dark state, there are only two parameters (the space is **two** dimensional, **α** and **θ**)

$$\Phi_1^A \;=\; \left(e^{i\alpha}\cos(\theta), 0, -\sin(\theta), 0\right)$$

- Only **one** non-zero component of the **vector potential**

$$A_\alpha = i\langle\Phi_1^A|\partial_\alpha|\Phi_1^A\rangle = -\cos^2\theta \qquad\qquad A_\theta = 0$$

- The corresponding **field** is

$$F_{\alpha\theta} = \partial_\alpha A_\theta - \partial_\theta A_\alpha = -\sin(2\theta)$$

Non-Abelian Potential Field

- In the **non**-Abelian case, there are two degenerate dark states, and there are three parameters (the space is 3-dimensional , **α, ϕ** and **θ**)

$$\Phi_1^N = \left(e^{i\alpha}\cos(\theta), 0, -\sin(\theta), 0\right)$$
$$\Phi_2^N = \left(e^{i\alpha}\sin(\phi)\sin(\theta), -\frac{1}{\sqrt{2}}\cos(\phi), \cos(\theta)\sin(\phi), \frac{1}{\sqrt{2}}\cos(\phi)\right)$$

- The components of the vector potential $A_{\mu ij} = i\langle\Phi_i^N|\partial_\mu|\Phi_j^N\rangle$

$$A_\theta = -\sin\phi\ \sigma_y, \qquad A_\phi = 0$$
$$A_\alpha = -\frac{1}{2}\sin\phi\sin(2\theta)\ \sigma_x - \cos^2\theta\ \sigma_\uparrow - \sin^2\phi\sin^2\theta\ \sigma_\downarrow$$

- The components of the field are $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$

Non-zero commutator

$$\partial_\theta A_\phi - \partial_\phi A_\theta = -\cos\phi\ \sigma_y$$
$$\partial_\theta A_\alpha - \partial_\alpha A_\theta = -\sin\phi\cos(2\theta)\sigma_x + \sin(2\theta)[\sigma_\uparrow - \sin^2\phi\ \sigma_\downarrow]$$
$$-i[A_\theta, A_\alpha] = \sin\phi\left(\cos^2\theta - \sin^2\phi\sin^2\theta\right)\sigma_x - \sin^2\phi\sin(2\theta)\sigma_z$$

What is really non-Abelian?

Necessity for Wilson loops

What is non-Abelian?

- There are a lot of misconceptions in the context of synthetic gauge fields about how to identify what is really non-Abelian.

Even published papers contain erroneous conclusions about how to identify non-Abelian gauge fields

Confusion 1: Gauge Dependence

- Consider the degenerate case, but now set **α=0**, then the vector potential has only one non-zero component
- All commutators of the vector potentials are zero
- So the field should be Abelian $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ as claimed in some recent reviews and papers
  - ➢ *N. Goldman, G. Juzeliunas, P. Ohberg, I. B. Spielman ,*
    Reports on Progress in Physics 77, 126401 (2014)
  - ➢ *J. Ruseckas, G. Juzeliunas, P. Ohberg ,G. M. Fleischhauer,*
    PRL 95, 010404 (2005)
- Yet, since there are two dark states, some called this case non-Abelian
  - ➢ *R. G. Unanyan, B. W. Shore, K. Bergmann et al PRA 59 2910 (1999)*

Potential and Field are not Adequate

- The vector potential is **not gauge invariant** $A \rightarrow UAU^\dagger - i(dU)U^\dagger$
- So, a decision about whether the system is Abelian or non-Abelian cannot be based on the vector potential

- The field is gauge **covariant**, $F = dA - iA^2 \rightarrow UFU^\dagger$
- … only because of cancellation of terms from curl & commutator

$$dA \rightarrow UdAU^\dagger - UAdU^\dagger + dUAU^\dagger + idUdU^\dagger$$

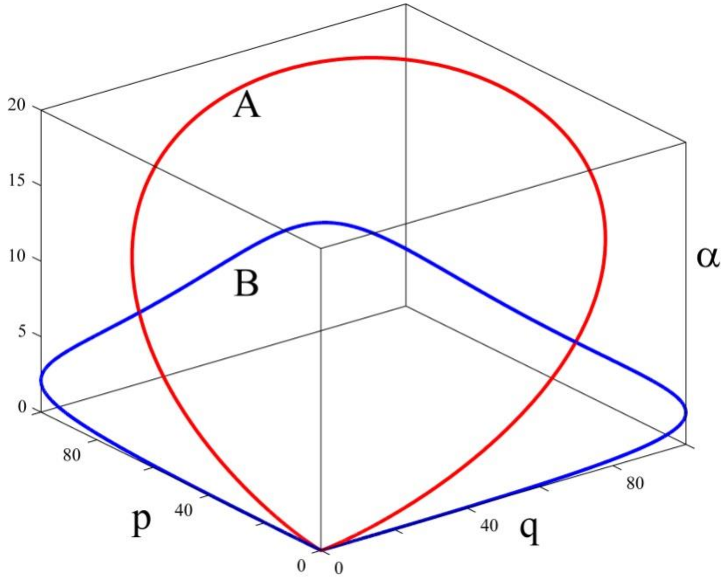
$$-iA^2 \rightarrow -iUA^2U^\dagger + UAdU^\dagger - dUAU^\dagger - idUdU^\dagger$$

- The field is **not gauge invariant** either

- Moreover, when you measure the field, it is challenging to distinguish **which** contributions **are from the commutator**!

Need for Wilson Loops

- Wilson Loops **are gauge invariant** and provide a reliable way to identify non-Abelian cases



- Evolve the system through two separate paths **A** and **B** in parameter space, evaluate Wilson loop at the end

- Switch their order, **B-A**, evolve again and measure the Wilson loop

- If the Wilson loops are the **same** in both orders then it is **Abelian**, if they are **different** then it is **non-Abelian**

Confusion 2: Two Loops are Not Enough

- In recent review, it was stated two Wilson loops will do the job
  - *N. Goldman, G. Juzeliunas, P. Ohberg, I. B. Spielman ,*
    Reports on Progress in Physics 77, 126401 (2014)

- Actually at least **three** Wilson loops are needed

- Because it is always true that ***Trace[AB]=Trace[BA]***

- So, we need NON-CYCLIC permutation of at least **THREE LOOPs:** A-B-C and then A-C-B

Confusion 3: Magnitude of Trace

- In another recent paper, for a U(2) gauge it was suggested that **W<2** is sufficient to identify non-Abelian
  - *N. Goldman, A. Kubasiak, P. Gaspartd, and M. Lewenstein PRA 79 023624 (2009)*
- That is incorrect, because, for the case when α=0, we get **W<2**
- But it is Abelian: Commutators vanish, and also we can show that Wilson loop value in order ABC and ACB are identical

Confusion 4: Only Expectations Matter

- The latest consensus by experts is that the system is non-Abelian if the Wilson matrices (not the trace) do not commute

$$[W_\circ(A), W_\circ(B)] \neq 0$$

- But U(2) matrices with the same trace are unitary similar

$$UW_\circ(A)W_\circ(B)U^\dagger = W_\circ(B)W_\circ(A)$$

- So even if the operators do not commute, in general their trace may

$$\mathrm{tr}\{\rho W_\circ(B)W_\circ(A)\} = \mathrm{tr}\{U^\dagger \rho U W_\circ(A)W_\circ(B)\}$$

since for mixed states, $\rho \equiv U^\dagger \rho U$

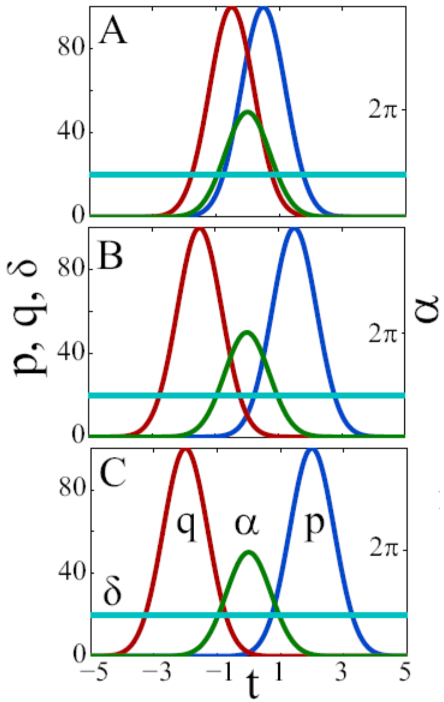
A way to measure
the full Wilson matrix

Creating the Closed Paths

- Keep detuning δ constant
- Vary the couplings and the phase in Gaussian pulses

$$h(t) = h_0 e^{-(t-\tau_h)^2/\sigma_h^2} \qquad h \in \{p, q, \alpha\}$$

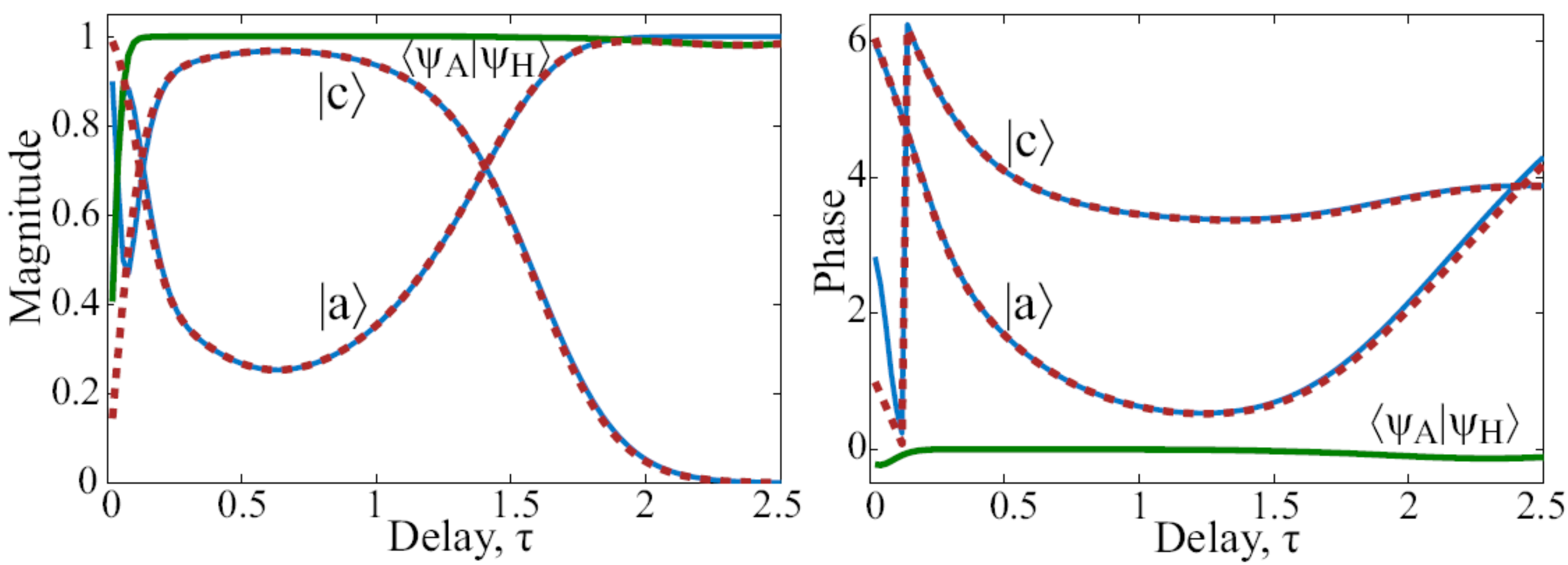
Change loops by changing delay, τ_h



Confirm Adiabatic Evolution

- Compare evolution by the full Hamiltonian with evolution within the two state subspace by the Wilson Matrix $d_t W = iAW$

$$H = \frac{\hbar}{2}\begin{pmatrix} 0 & e^{i\alpha}p & 0 & e^{i\alpha}p \\ e^{-i\alpha}p & \delta & q & 0 \\ 0 & q & 0 & q \\ e^{-i\alpha}p & 0 & q & \mp\delta \end{pmatrix}$$

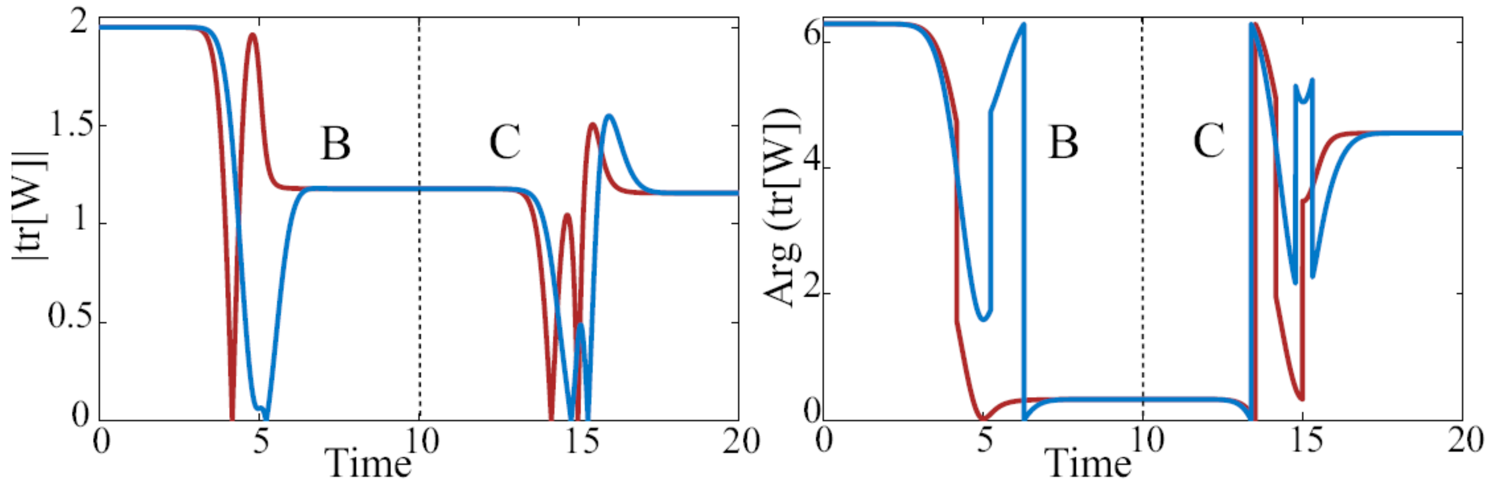


Gauge invariance of Wilson Loops

- Compare two Wilson loops that differ by a gauge transformation

$$U^\dagger = ((1,0),(0,e^{i\zeta(t)})) \text{ and } \zeta(t) = \frac{1}{2}\alpha(t)$$

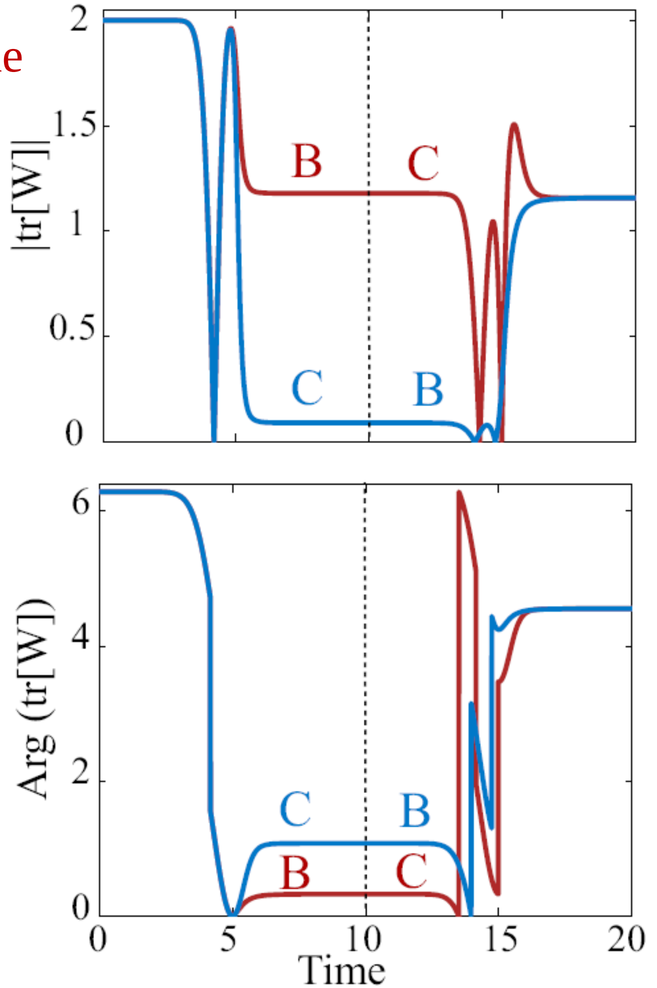
- They have the same final magnitude and phase



Two loops versus Three loops

Magnitude

Phase



Measuring Wilson Loop

- The Wilson Matrix can be factorized as

$$W_\circ = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$$

- The Wilson loop is the trace $w_{11} + w_{22} = 2\cos\theta\cos(\vartheta_1)e^{i\vartheta}$
- Note the phase comes from the U(1) part, so it always commutes

- Starting with Φ_1^N gives $w_{11} = e^{i(\vartheta+\vartheta_1)}\cos\theta$ and $w_{21} = e^{i(\vartheta-\vartheta_2)}\cos\theta$

- That only gives us $2\vartheta + \vartheta_1 - \vartheta_2$ and $\vartheta_1 + \vartheta_2$

- We will also need $\vartheta_1 - \vartheta_2$

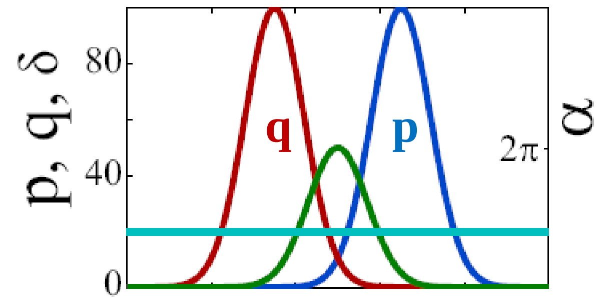
Measuring the Matrix Elements

- Start with all the population in bare state $|1\rangle$, then like in STIRAP vary the couplings counter-intuitively, **q** precedes **p**

$$p = \frac{1}{\sqrt{2}}\Omega \boxed{\sin(\theta)} \cos(\phi) \qquad q = \frac{1}{\sqrt{2}}\Omega \boxed{\cos(\theta)} \cos(\phi)$$

- At the start the wavefunction matches the first dark state

$$\Phi_1^N = (e^{i\alpha}\boxed{\cos(\theta)}, 0, \boxed{-\sin(\theta)}, 0)$$



at the end, the population is in bare state $|\mathbf{3}\rangle$

(1,0,0,0) → (0,0, -1 ,0)

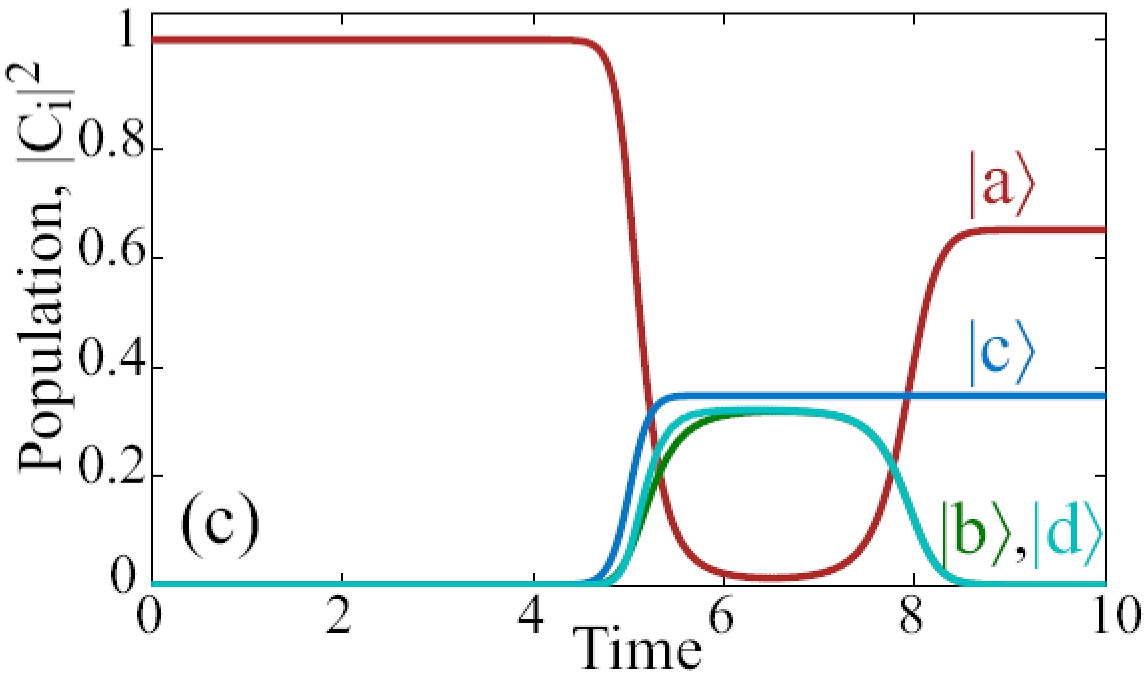
- But the coupling due to constant detuning

$\delta = \Omega \sin(\phi)$ transfers population to the other dark state as well

$$\Phi_2^N = \left(e^{i\alpha}\sin(\phi)\boxed{\sin(\theta)}, -\frac{1}{\sqrt{2}}\cos(\phi), \cos(\theta)\sin(\phi), \frac{1}{\sqrt{2}}\cos(\phi)\right)$$

leading to population in bare state $|\mathbf{1}\rangle$ at the end (0,0,0,0) → (1,0,0,0)

Counterintuitive Evolution



The state evolution matches first column of Wilson matrix

$$\Psi^F(T) = (0.626 + 0.510i, 0, -0.570 - 0.152i, 0)$$

$$W_\circ^F(T) = \begin{pmatrix} 0.570 + 0.155i & -0.806 + 0.025i \\ 0.622 + 0.513i & 0.546 + 0.227i \end{pmatrix}$$

Still Not Enough

- That only gives us $2\vartheta + \vartheta_1 - \vartheta_2$ and $\vartheta_1 + \vartheta_2$
- We **Still** need $\vartheta_1 - \vartheta_2$

Run it Backwards

- Again start with all the population in bare state $|1\rangle$

$$p = \frac{1}{\sqrt{2}}\Omega \boxed{\sin(\theta)} \cos(\phi) \qquad q = \frac{1}{\sqrt{2}}\Omega \cos(\theta)\cos(\phi)$$

- But, now do intuitive evolution, **p** precedes **q**

  so now the initial state matches the second dark state

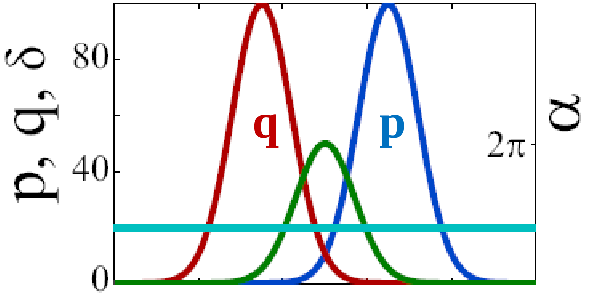
$$\Phi_2^N = \left(e^{i\alpha}\sin(\phi)\boxed{\sin(\theta)}, -\frac{1}{\sqrt{2}}\cos(\phi), \cos(\theta)\sin(\phi), \frac{1}{\sqrt{2}}\cos(\phi)\right)$$

  which evolves as (1,0,0,0) → (0,0,1,0)

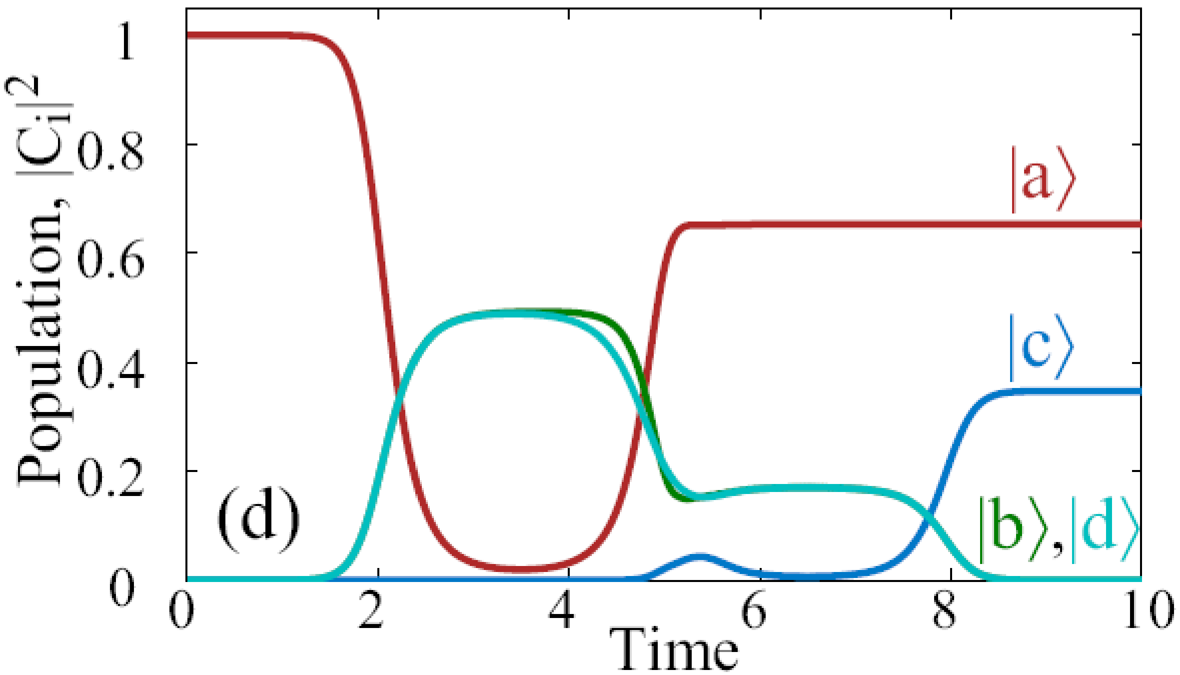
- Due to the coupling there is population in the first dark state

$$\Phi_1^N = \left(e^{i\alpha}\cos(\theta), 0, -\sin(\theta), 0\right)$$

  which evolves as (0,0,0,0) → (1,0,0,0)



Intuitive Evolution



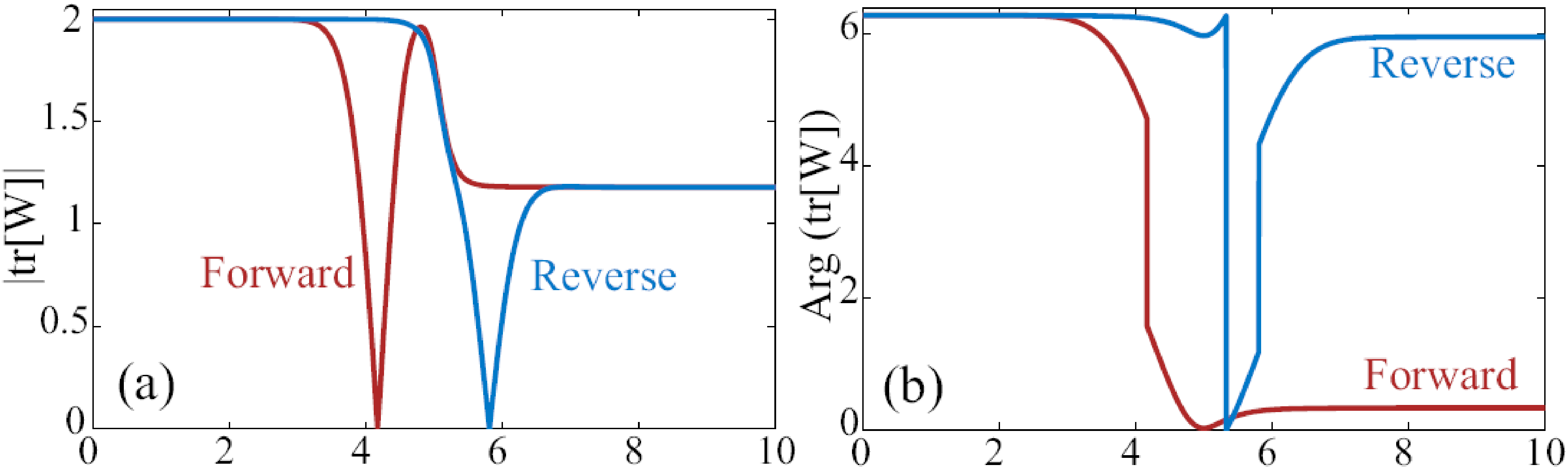
The state evolution matches second column of Wilson matrix

$$\Psi^R(T) \;=\; (0.626 - 0.510i, 0, 0.544 - 0.226i, 0)$$

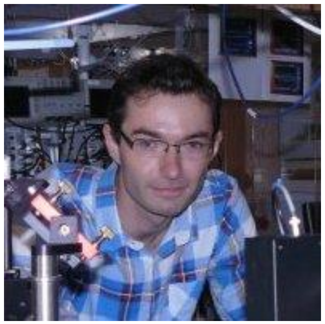
$$W_\circ^R(T) \;=\; \begin{pmatrix} 0.570 - 0.155i & 0.622 - 0.513i \\ -0.806 - 0.025i & 0.546 - 0.227i \end{pmatrix}$$

Forward and Backward

Magnitude and Phase of the Wilson loop, for both cases



Thank You!



- Miroslav Gajdacz, *University of Aarhus, Denmark*
  currently *Development Engineer at OFS Denmark*
  Simulations for external states implementation

- Support from National Science Foundation (NSF) under Grants No. PHY-1313871 and PHY-1707878

- Kavli Institute of Theoretical Physics at the University of California, Santa Barbara where this work was initiated under a Kavli Scholarship.