DEPARTMENT OF PHYSICS & ASTRONOMY - PHYSICS

CMQC Center for Macroscpic Quantum Control @ SNU

Practical Probes and Measurements for Phase estimation in lossy environment

Changhun Oh

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Contents

Background and Motivation

- Optical interferometry is a conventional tool of phase estimation.
  e.g.) Mach-Zehnder interferometer, Michelson interferometer …

- Various nonclassical states have been proposed for quantum enhanced phase estimation in optical interferometry.
  e.g.) Coherent & Squeezed vacuum, NOON state, Two-mode squeezed vacuum, twin Fock state, entangled coherent state …

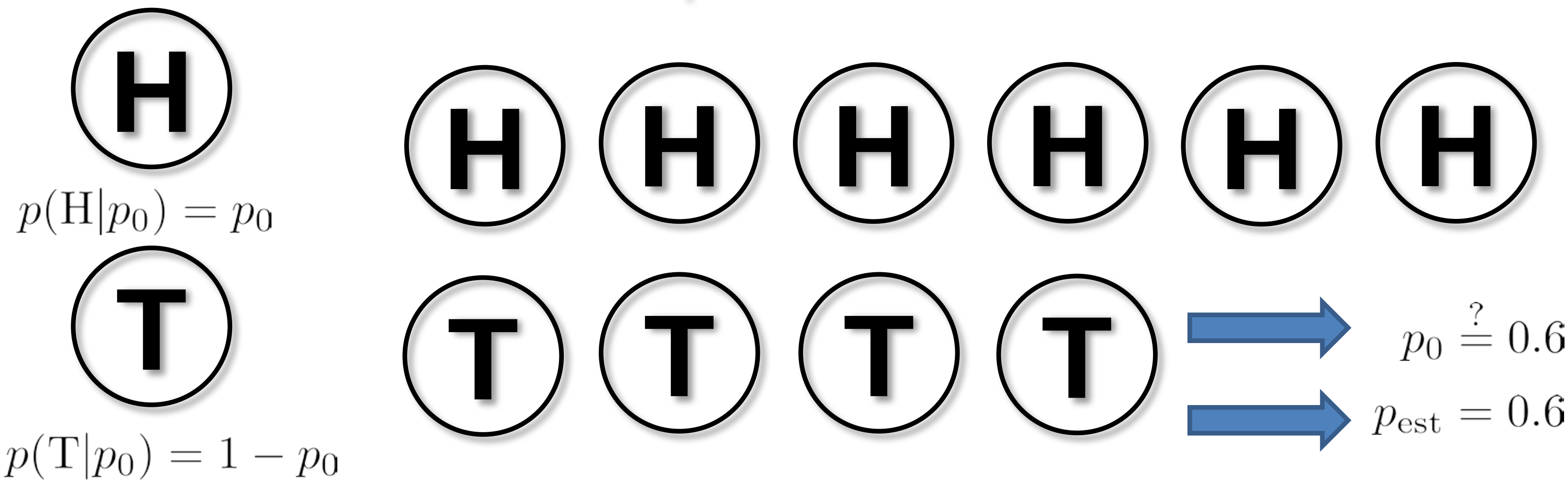
- We focus on practical quantum states which require only a Laser and nonlinear medium (squeezing) among various proposed states in lossy Mach-Zehnder interferometer.

- We consider parity detection and homodyne detection for practical measurement setup.

Classical Parameter Estimation

$$\phi \xrightarrow{p(i|\phi)} i \in \{1, 2, ..., D\}$$

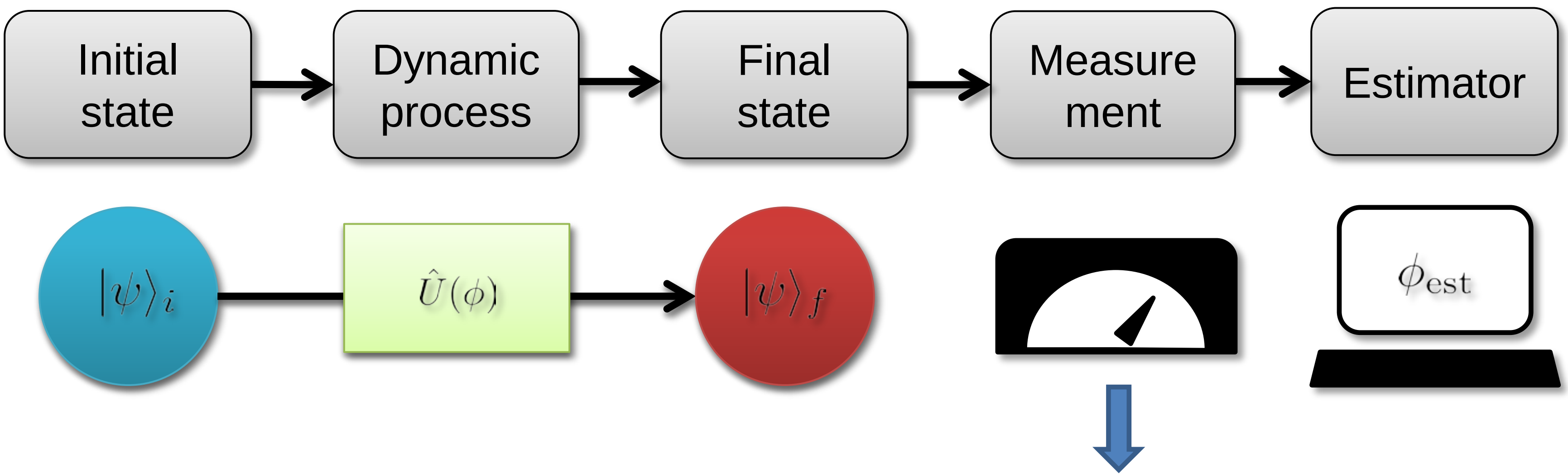
$$\sum_i p(i|\phi) = 1$$



The goal is to estimate ϕ as accurate as possible.

$\Rightarrow$ Minimize $\Delta^2\phi \equiv \langle(\phi - \phi_{\mathrm{est}})^2\rangle$

Quantum Parameter Estimation



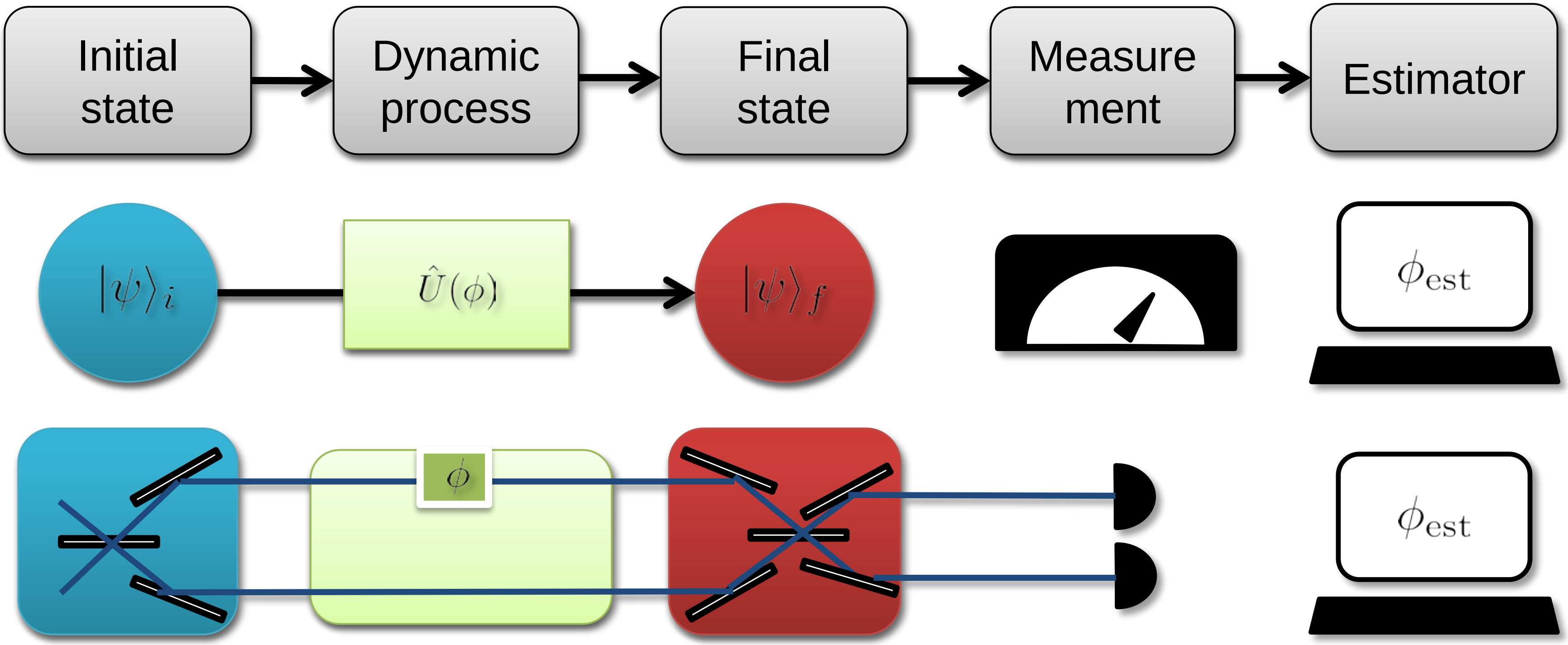
$$p(i|\phi) = \mathrm{Tr}[\hat{\Pi}_i |\psi\rangle\langle\psi|_f]$$

The goal is to estimate ϕ as accurate as possible.

⇨ Minimize $\Delta^2\phi \equiv \langle(\phi - \phi_{\text{est}})^2\rangle$

1. Input state
2. Measurement setup

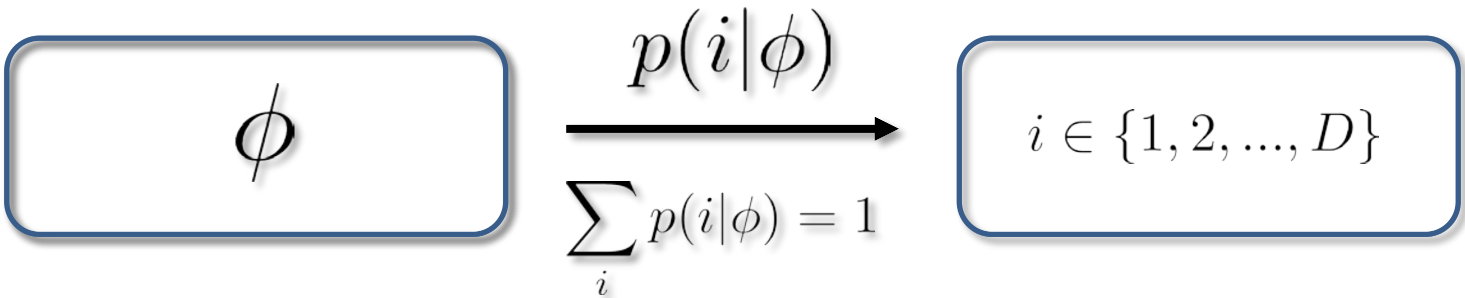
Quantum Parameter Estimation



General algorithm to estimate an unknown parameter ϕ.
(e.g. gravitational wave detector, frequency estimation…)

B. M. Escher *et al.*, Nat. Phys. **7**, 406 (2011)

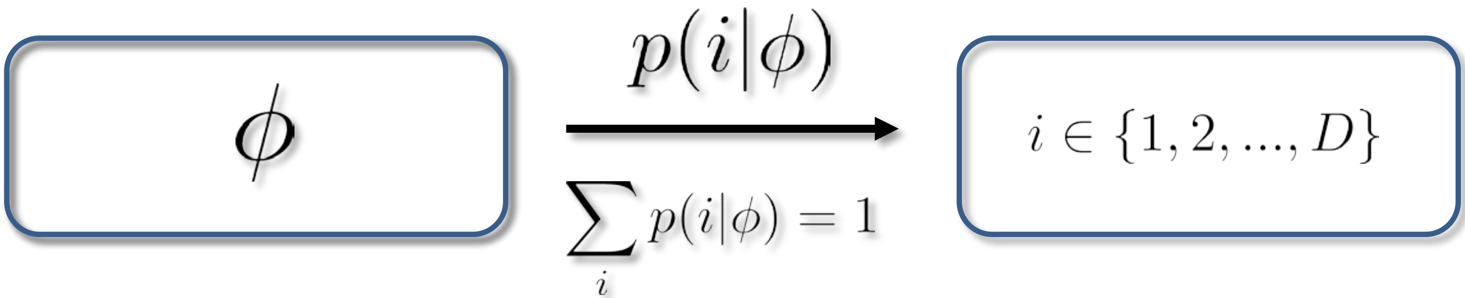
Classical Cramer-Rao Inequality



$$\Delta^2\phi \geq \quad ?$$

H. Cramér, *Mathematical Methods of Statistics* (1946).

Classical Cramer-Rao Inequality



Classical Cramer-Rao inequality :

$$\Delta^2\phi \geq \frac{1}{MF_C(\phi)},$$

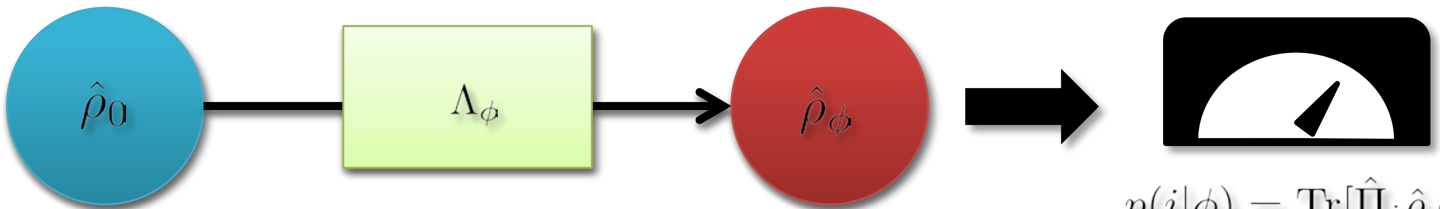
Classical Fisher information :

$$F_C(\phi) = \sum_i \frac{1}{p(i|\phi)} \left(\frac{\partial p(i|\phi)}{\partial \phi} \right)^2.$$

The inequality is asymptotically saturable using maximum likelihood estimator.

H. Cramér, *Mathematical Methods of Statistics* (1946).

Quantum Cramer-Rao Inequality



$$\sum_i \hat{\Pi}_i = \hat{1}$$

$$p(i|\phi) = \mathrm{Tr}[\hat{\Pi}_i \hat{\rho}_\phi]$$

Classical Cramer-Rao inequality : $F_C(\phi) = \sum_i \frac{1}{p(i|\phi)} \left(\frac{\partial p(i|\phi)}{\partial \phi} \right)^2.$

Quantum Cramer-Rao inequality :

$$\Delta^2 \phi \equiv \langle (\phi - \phi_{\mathrm{est}})^2 \rangle \geq \frac{1}{M F_Q(\hat{\rho}_0, \Lambda_\phi)}, \qquad F_Q(\hat{\rho}_0, \Lambda_\phi) = \max_{\{\hat{\Pi}_i\}} F_C(\hat{\rho}_0, \Lambda_\phi).$$

Still, we have a freedom to choose an input state.

S. L. Braunstein *et al.* PRL **72**, 3439 (1994)

Quantum Cramer-Rao Inequality

Quantum Cramer-Rao inequality :

$$\Delta^2\phi \equiv \langle(\phi - \phi_{\text{est}})^2\rangle \geq \frac{1}{MF_Q(\hat{\rho}_0, \Lambda_\phi)},$$

where $F_Q(\hat{\rho}_0, \Lambda_\phi)$ is quantum Fisher information of $\hat{\rho}_0$ with respect to Λ_ϕ .

$$F_Q(\hat{\rho}_0) = \mathrm{Tr}[\hat{\rho}_\phi \hat{L}_\phi^2]$$
$$= 2\sum_{n,m} \frac{|\langle\psi_m|\partial_\phi\hat{\rho}_\phi|\psi_n\rangle|^2}{p_n + p_m}$$

$$\frac{\partial\hat{\rho}_\phi}{\partial\phi} = \frac{\hat{\rho}_\phi \hat{L}_\phi + \hat{L}_\phi \hat{\rho}_\phi}{2}.$$

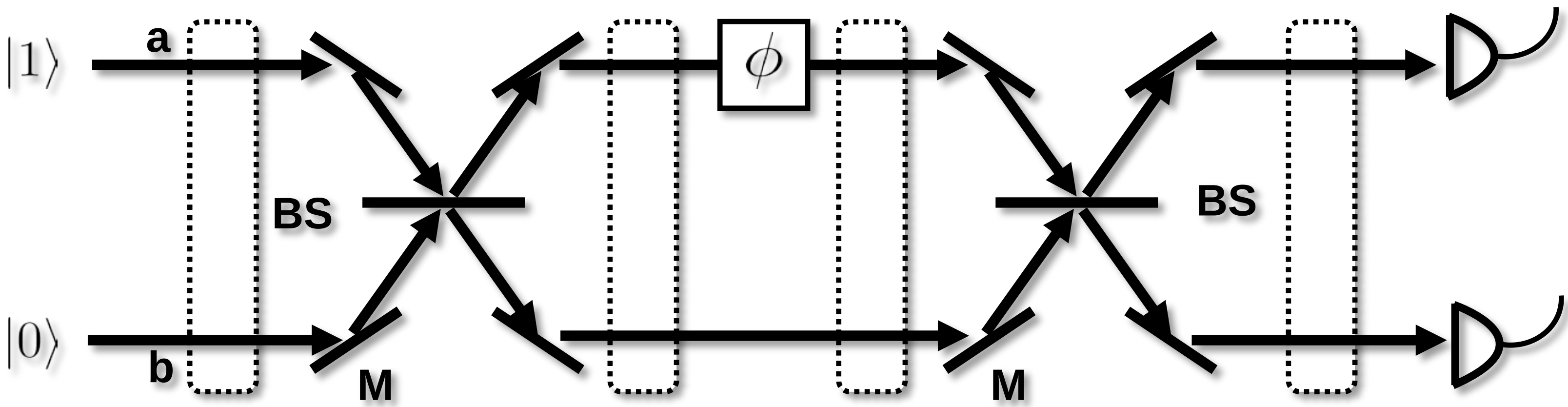
$$\hat{\rho}_\phi = \sum_n p_n |\psi_n\rangle\langle\psi_n|.$$

Quantum Fisher information gives the optimal sensitivity we can achieve with a given input state $\hat{\rho}_0$ and a quantum channel Λ_ϕ .

- It is still challenging to find the optimal measurement setup for a given state.
- Even when we found the setup, the implementation of the setup is difficult in general.

M. G. A. Paris, Int. J. Quantum Inf. **7**, 125 (2009)

Mach-Zehnder Interferometer



$|1\rangle_a|0\rangle_b$ $\quad |0\rangle_a|1\rangle_b + |1\rangle_a|0\rangle_b$ $\quad |0\rangle_a|1\rangle_b + e^{-i\phi}|1\rangle_a|0\rangle_b$ $\quad \cos\frac{\phi}{2}|0\rangle_a|1\rangle_b + \sin\frac{\phi}{2}|1\rangle_a|0\rangle_b$

$$p(10|\phi) = \sin^2\frac{\phi}{2}$$

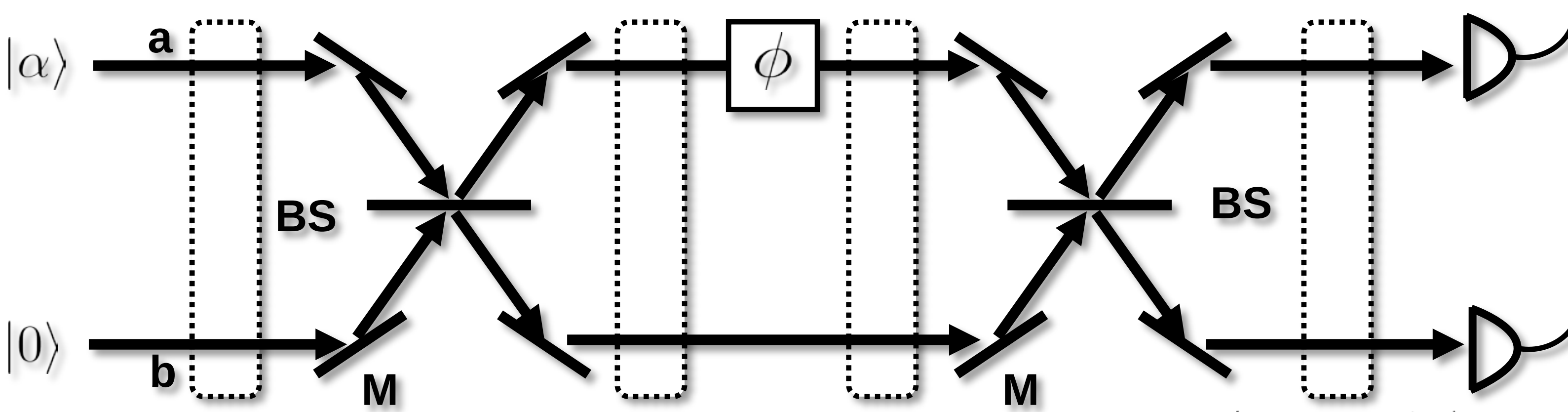
$$p(01|\phi) = \cos^2\frac{\phi}{2}$$

$$F_C(\phi) = \frac{1}{\sin^2\frac{\phi}{2}}\left(\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right)^2 + \frac{1}{\cos^2\frac{\phi}{2}}\left(-\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right)^2 = 1$$

$$F_Q(\phi) = 1$$

V. Giovannetti *et al.*, Science **306**, 1330 (2004).

Mach-Zehnder Interferometer



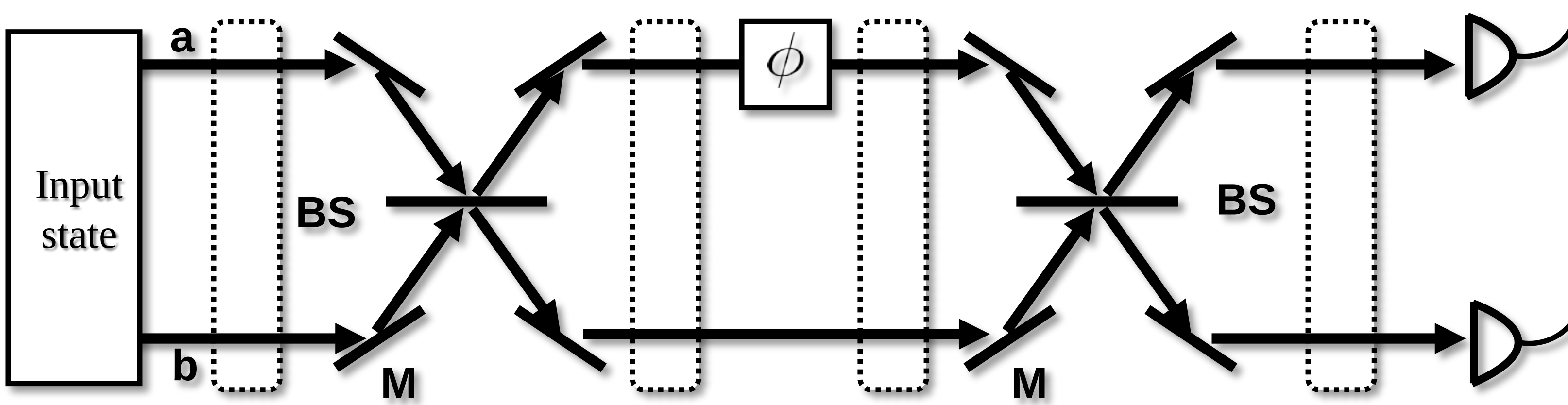
$$|\alpha\rangle_a|0\rangle_b \qquad \left|\frac{\alpha}{\sqrt{2}}\right\rangle_a\left|\frac{\alpha}{\sqrt{2}}\right\rangle_b \qquad \left|\frac{\alpha e^{-i\phi}}{\sqrt{2}}\right\rangle_a\left|\frac{\alpha}{\sqrt{2}}\right\rangle_b \qquad \left|\alpha\sin\frac{\phi}{2}\right\rangle\left|\alpha\cos\frac{\phi}{2}\right\rangle$$

$$F_Q(\phi) = 2\alpha^2 \propto \bar{n}$$

Shot-noise limit
(Classical limit)

Can we improve the precision of estimation by using nonclassical resources?

Mach-Zehnder Interferometer



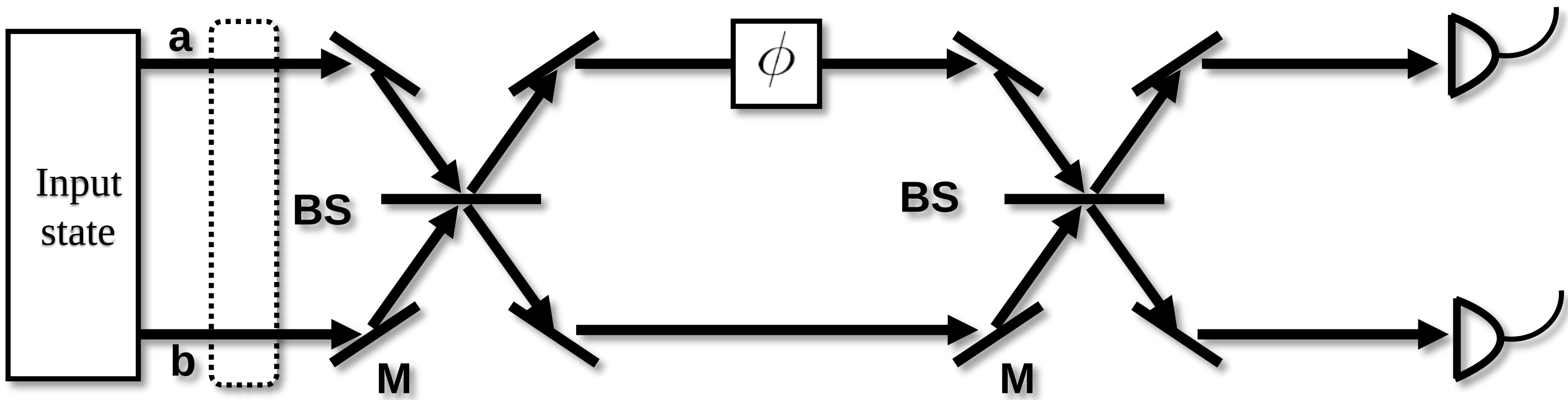
$|N0\rangle + |0N\rangle$ $\quad$ $e^{-iN\phi}|N0\rangle + |0N\rangle$

$$F_Q(\phi) = N^2$$

→ Heisenberg limit
(Quantum enhancement)

J. P. Dowling, Contemp. Phys. **49**, 125 (2008).

Mach-Zehnder Interferometer



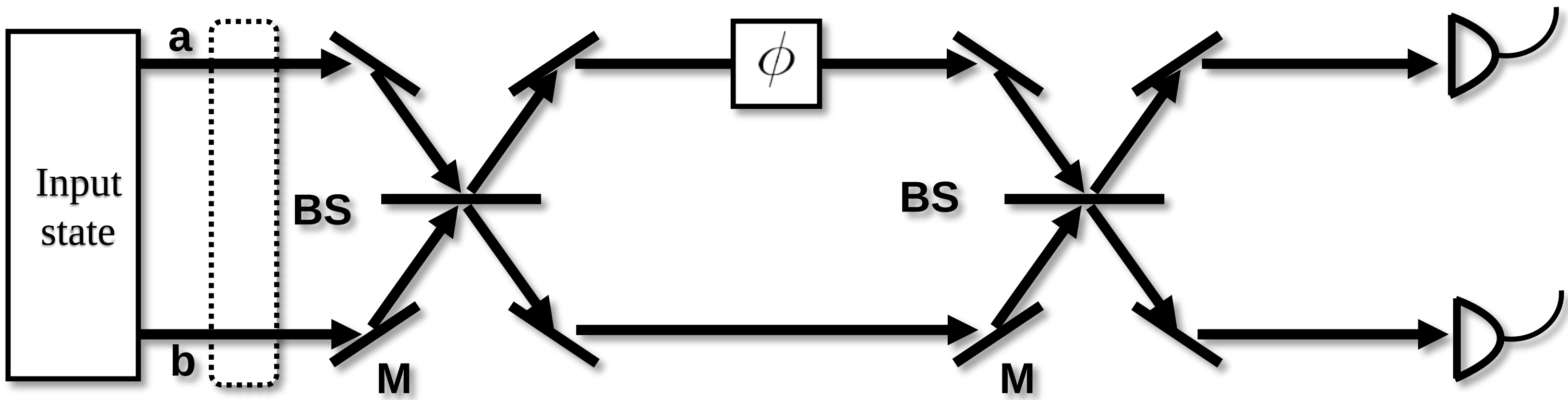
$$|\alpha\rangle|\xi\rangle = D_1(\alpha)S_2(\xi)|0\rangle$$

- Coherent & Squeezed vacuum
- Twin Fock state
- Two mode squeezed state
- NOON state
- Entangled coherent state

$$F_Q \propto \bar{n}^2$$

L. Pezzé and A. Smerzi, PRL **100**, 073601 (2008)

Mach-Zehnder Interferometer



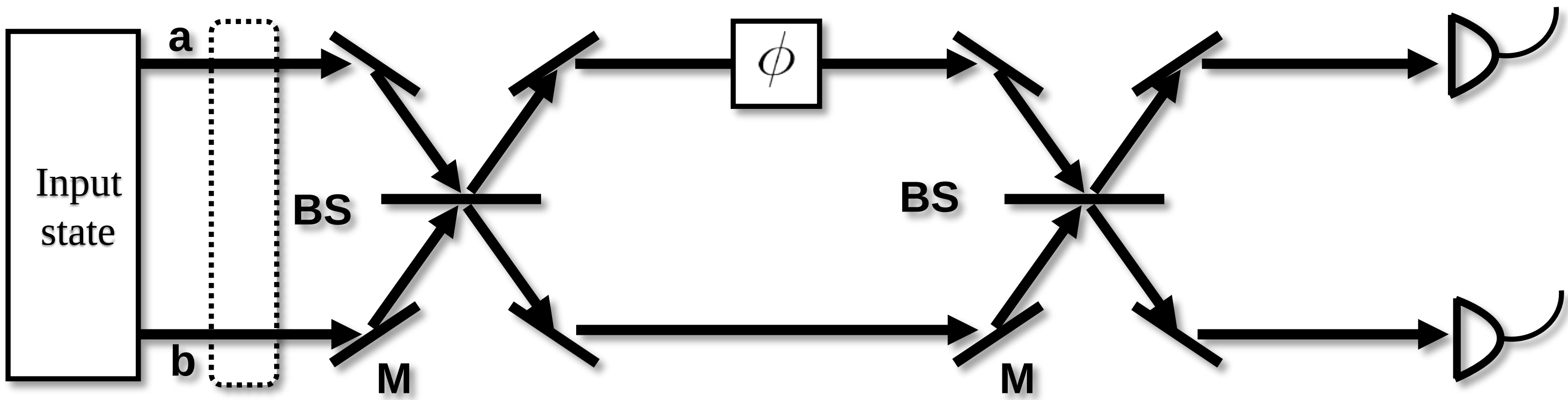
$|N\rangle|N\rangle$

- Coherent & Squeezed vacuum
- Twin Fock state
- Two mode squeezed state
- NOON state
- Entangled coherent state

$$F_Q \propto \bar{n}^2$$

M. J. Holland and K. Burnett, PRL **71**, 1355 (1993)

Mach-Zehnder Interferometer



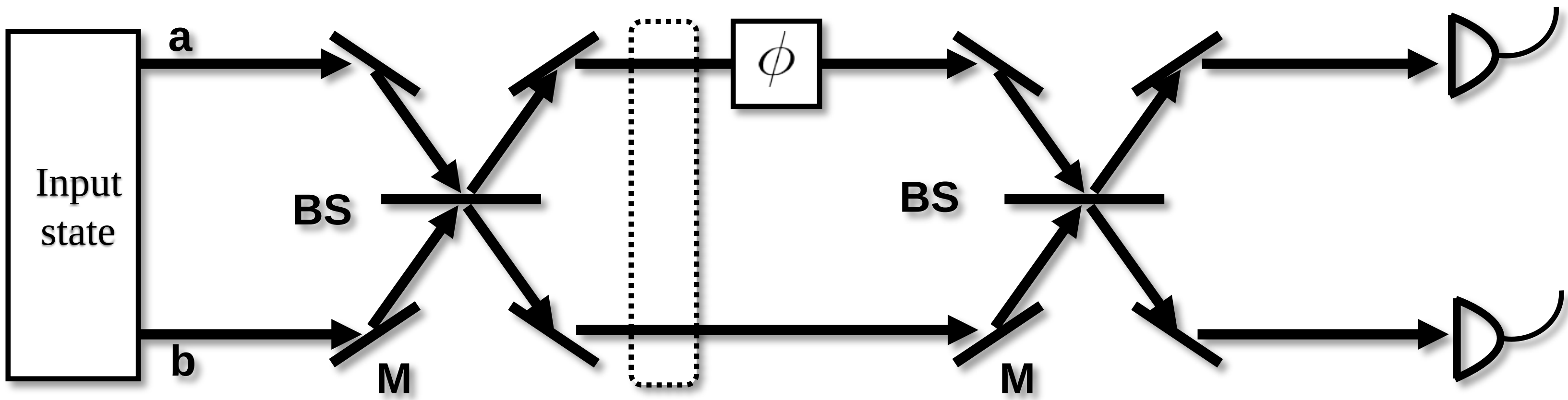
$$|\xi\rangle = S_{12}(\xi)|0\rangle$$

- Coherent & Squeezed vacuum
- Twin Fock state
- Two mode squeezed state
- NOON state
- Entangled coherent state

$$F_Q \propto \bar{n}^2$$

P. M. Anisimov, PRL **104**, 103602 (2010).

Mach-Zehnder Interferometer



$$|\xi\rangle = S_{12}(\xi)|0\rangle$$

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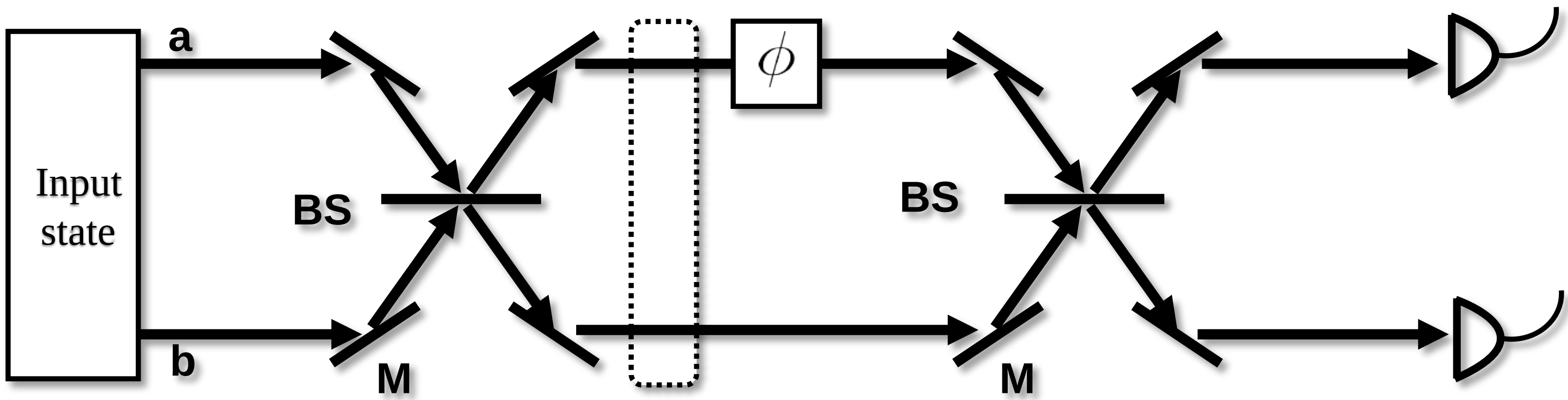
$$F_Q \propto \bar{n}^2$$

Mach-Zehnder Interferometer



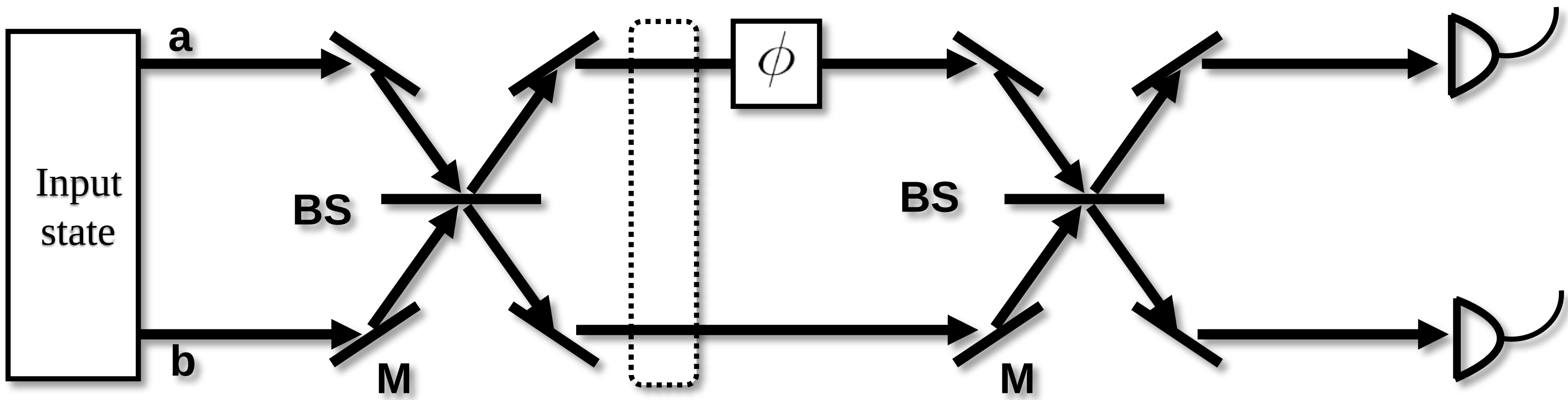
$$|N0\rangle + |0N\rangle$$

- Coherent & Squeezed vacuum
- Twin Fock state
- Two mode squeezed state
- NOON state
- Entangled coherent state

$$F_Q \propto \bar{n}^2$$

J. P. Dowling, Contemp. Phys. **49**, 125 (2008).

Mach-Zehnder Interferometer



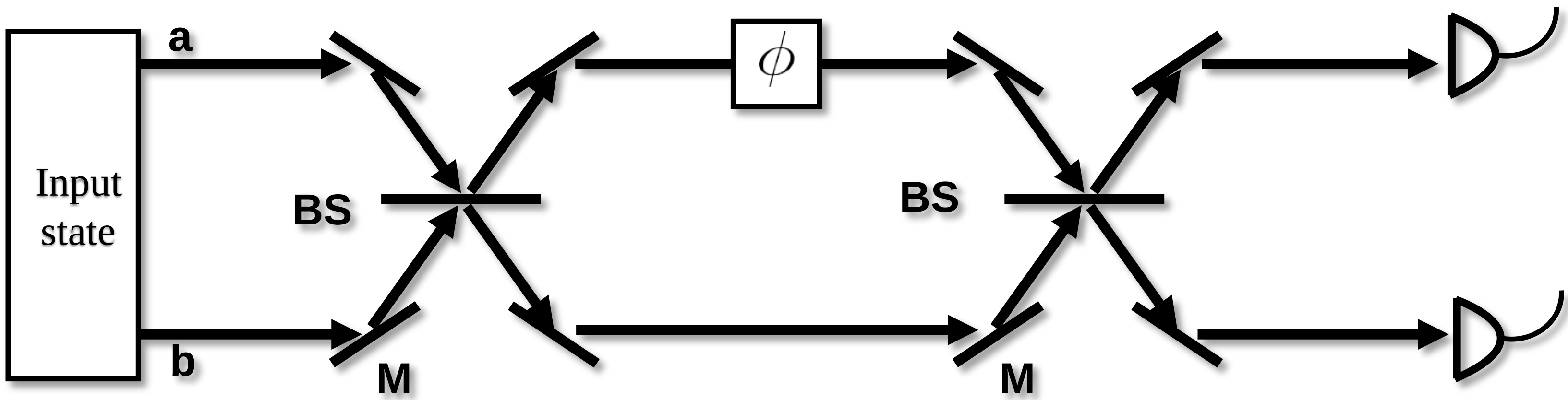
$$|\alpha\rangle|\alpha\rangle + |-\alpha\rangle|-\alpha\rangle$$

- Coherent & Squeezed vacuum
- Twin Fock state
- Two mode squeezed state
- NOON state
- Entangled coherent state

$$F_Q \propto \bar{n}^2$$

J. Joo et al. PRL **107**, 083601 (2011)

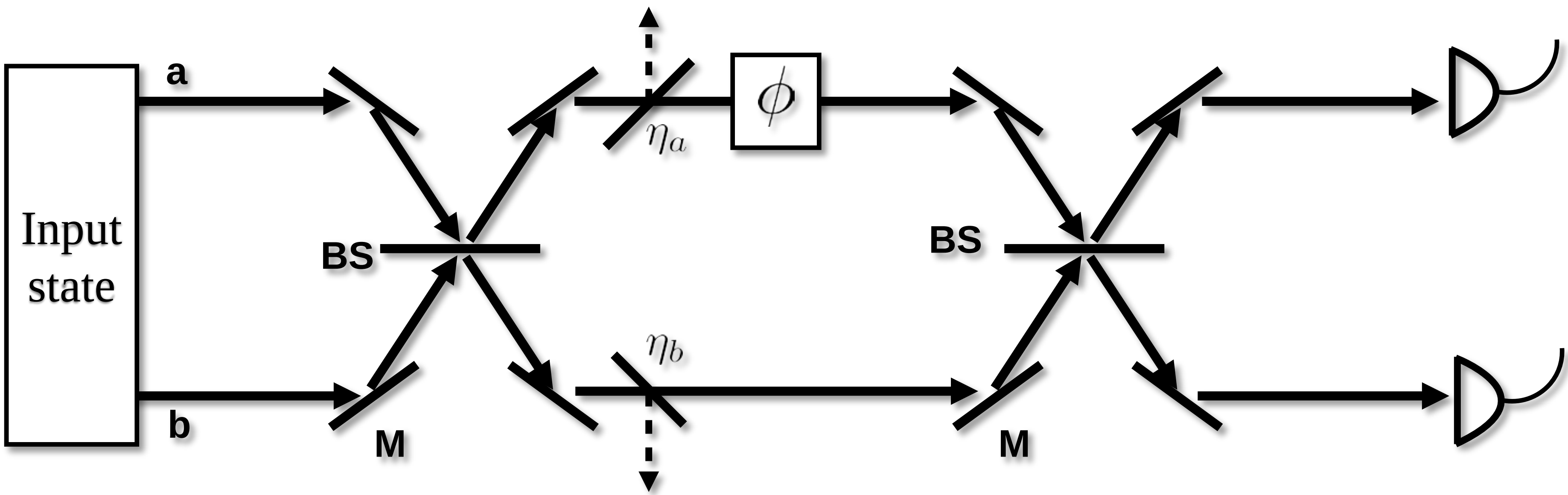
Mach-Zehnder Interferometer



- Coherent & Squeezed vacuum
- Twin Fock state
- Two mode squeezed state
- NOON state
- Entangled coherent state

Generated by a LASER and a nonlinear medium.
→ Practical resources.

Lossy Mach-Zehnder Interferometer



1. Symmetric losses : $\eta_a = \eta_b$
2. Sample loss : $\eta_a < 1, \eta_b = 1$

PRA **96**, 062304 (2017), C. Oh *et al.*

Quantum Fisher Information

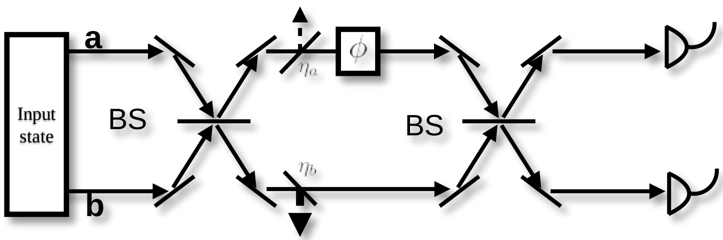
- Coherent & Squeezed vacuum

$$|\alpha\rangle|\xi\rangle = D_1(\alpha)S_2(\xi)|0\rangle$$

$$\bar{n} = \alpha^2 + \sinh^2 r$$

$$F_Q = \alpha^2 + \frac{\sinh^2 2r}{2} + \alpha^2 e^{2r} + \sinh^2 r \propto \bar{n}^2$$

1. Symmetric losses : $\eta_a = \eta_b$



2. Sample loss : $\eta_a < 1, \eta_b = 1$

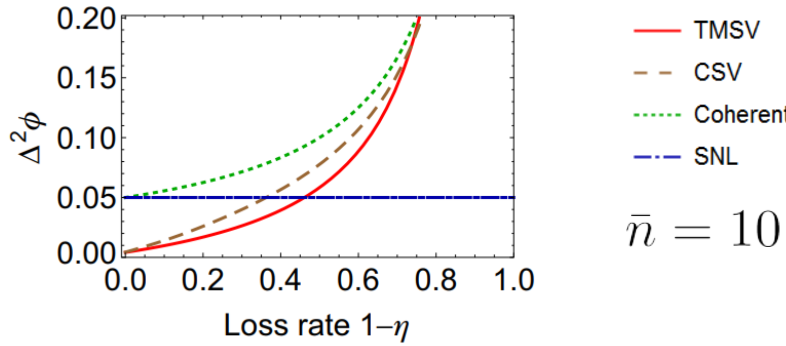
PRA **96**, 062304 (2017), C. Oh *et al.*

- Two mode squeezed state

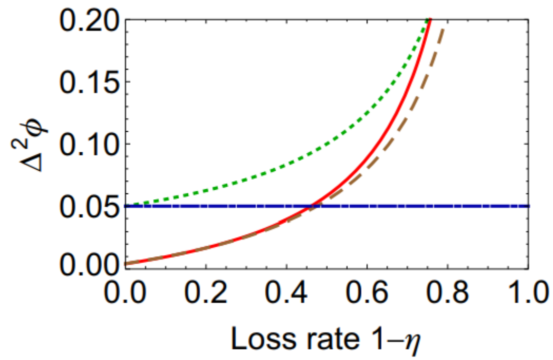
$$|\xi\rangle = S_{12}(\xi)|0\rangle$$

$$\bar{n} = 2\sinh^2 r$$

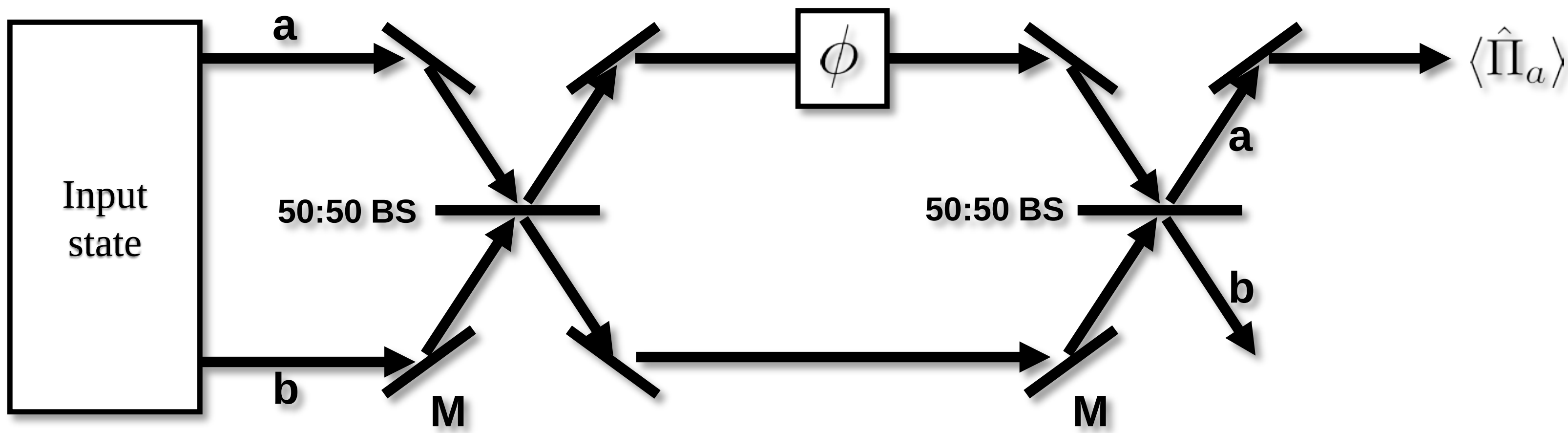
$$F_Q = \bar{n}(\bar{n}+2) \propto \bar{n}^2$$



$\bar{n} = 10$



Parity Detection

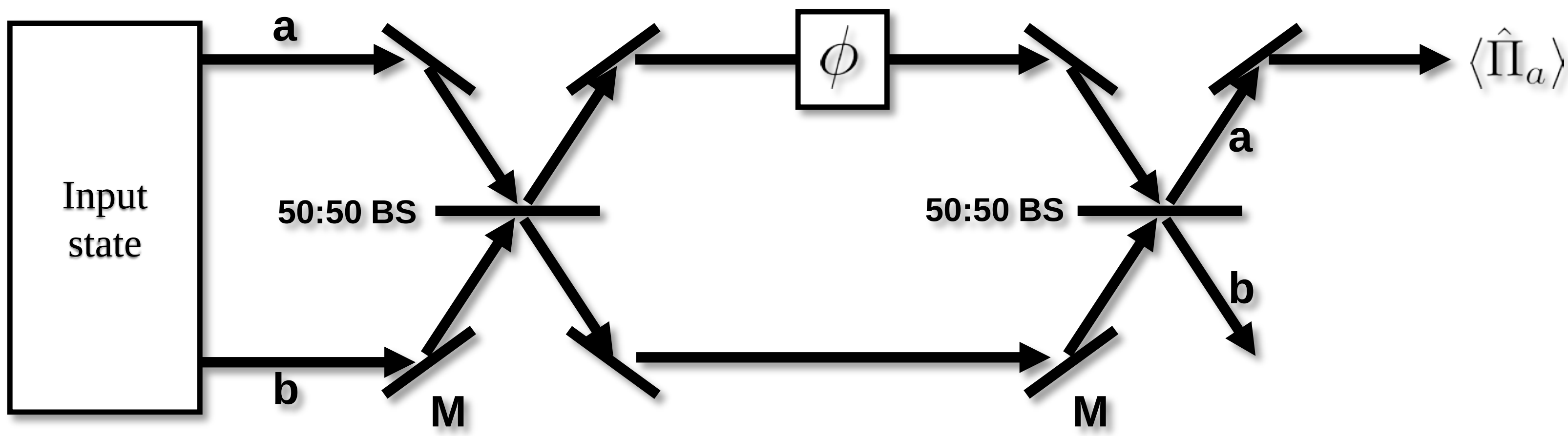


- Parity detection $\hat{\Pi}_a = (-1)^{\hat{N}_a}$

- It achieves Heisenberg limit for TMSV, CSV, and in general for all pure path symmetric states.

P. M. Anisimov *et al.*, PRL **104**, 103602 (2010).
K. P. Seshadreesan, P. M. Anisimov, H. Lee, and J. P. Dowling, NJP **13**, 083026 (2011).
K. P. Seshadreesan, S. Kim, J. P. Dowling, and H. Lee, PRA **87**, 043833 (2013).

Parity Detection



- Parity detection $\hat{\Pi}_a = (-1)^{\hat{N}_a}$

$$\Delta^2\phi = \frac{\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2}{(\partial \langle \hat{O} \rangle / \partial \phi)^2}$$

- Coherent & Squeezed vacuum

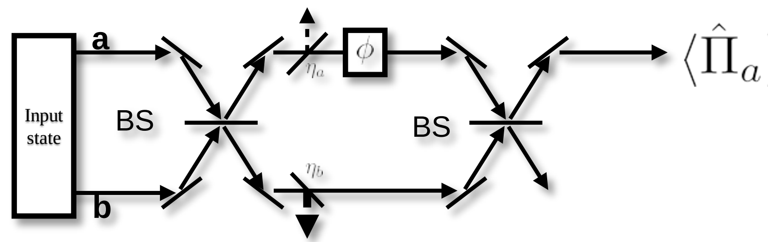
$$\Delta^2\phi = \frac{1}{\alpha^2 e^{2r} + \sinh^2 r} \sim \frac{1}{\bar{n}^2}$$

- Two-mode squeezed state

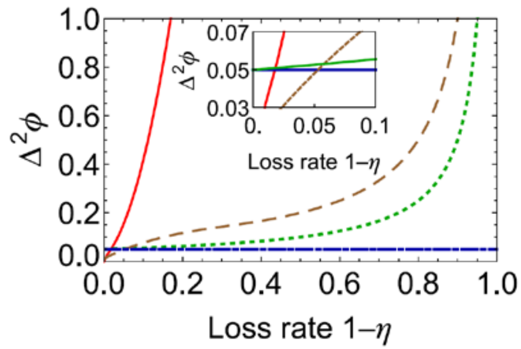
$$\Delta^2\phi = \frac{1}{2\sinh^2 2r} = \frac{1}{\bar{n}(\bar{n}+2)}$$

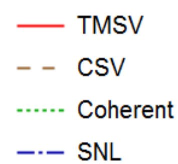
Parity Detection

- Parity detection $\hat{\Pi}_a = (-1)^{\hat{N}_a}$



1. Symmetric losses : $\eta_a = \eta_b$

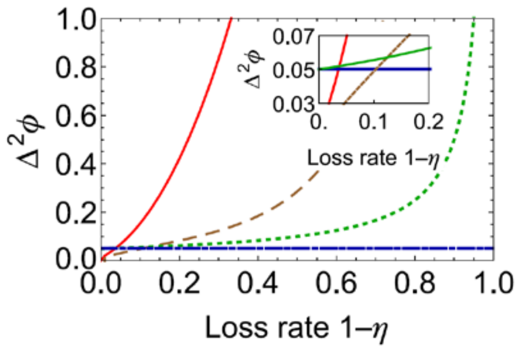




$\bar{n} = 10$

$$\Delta^2\phi = \frac{\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2}{(\partial \langle \hat{O} \rangle / \partial \phi)^2}$$

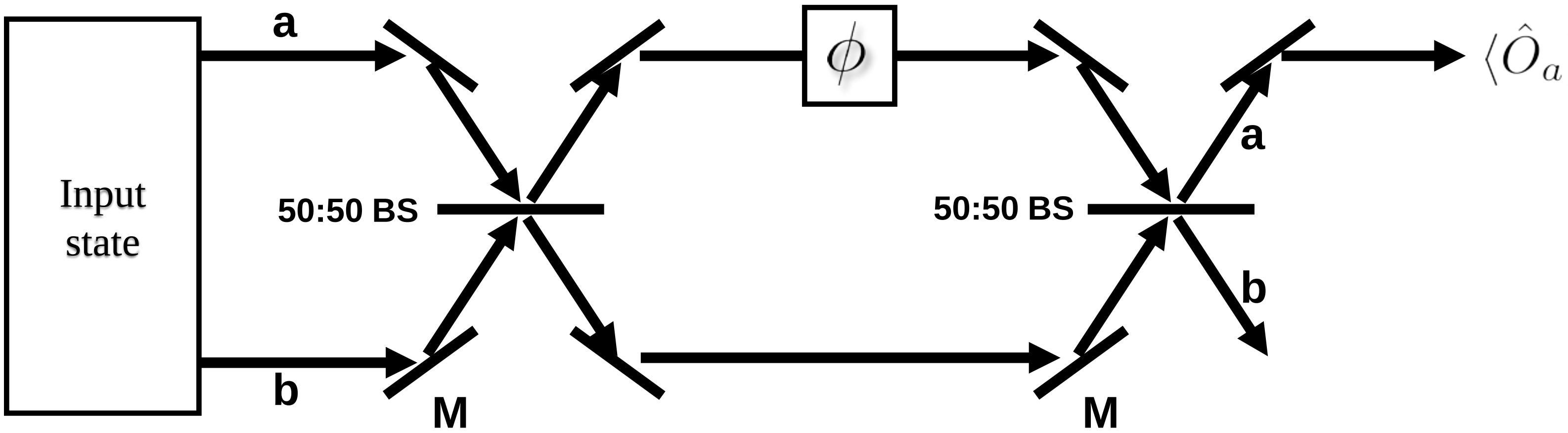
2. Sample loss : $\eta_a < 1, \eta_b = 1$



Parity detection is very fragile against photon loss.

PRA **96**, 062304 (2017), C. Oh *et al.*

Single Homodyne Detection



- Homodyne detection

$$\Delta^2\phi = \frac{\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2}{(\partial \langle \hat{O} \rangle / \partial \phi)^2}$$

- Coherent & Squeezed vacuum

$\hat{O} = \hat{P}_a$

$$\Delta^2\phi = \frac{1}{\alpha^2 e^{2r}} \sim \frac{1}{\bar{n}^2}$$

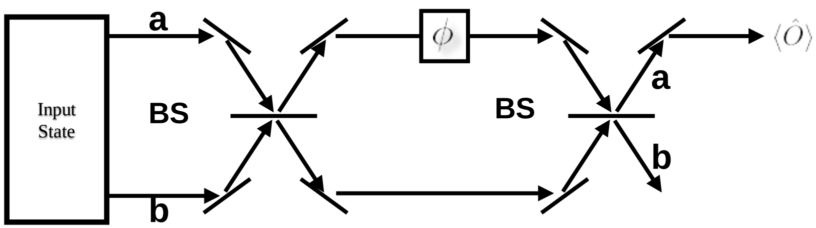
- Two mode squeezed state

$\hat{O} = \hat{X}_a^2$

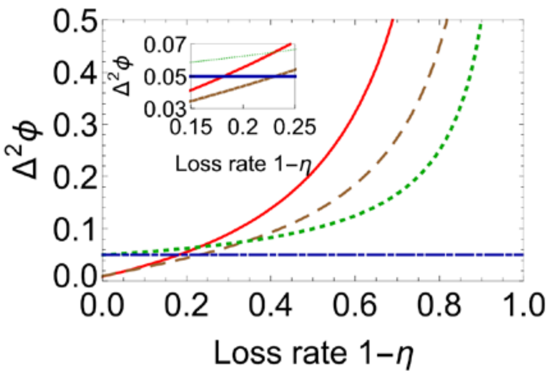
$$\Delta^2\phi = \frac{1}{2e^{2r}} \left(\frac{1}{\sinh^2 r} + \frac{1}{\cosh^2 r} \right) \sim \frac{1}{\bar{n}^2}$$

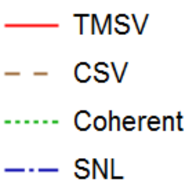
Single Homodyne Detection

- Homodyne detection



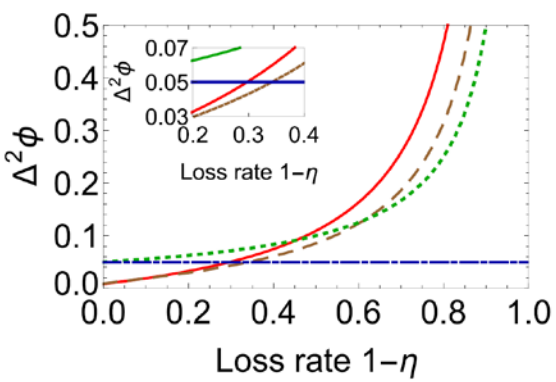
1. Symmetric losses : $\eta_a = \eta_b$





$\bar{n} = 10$

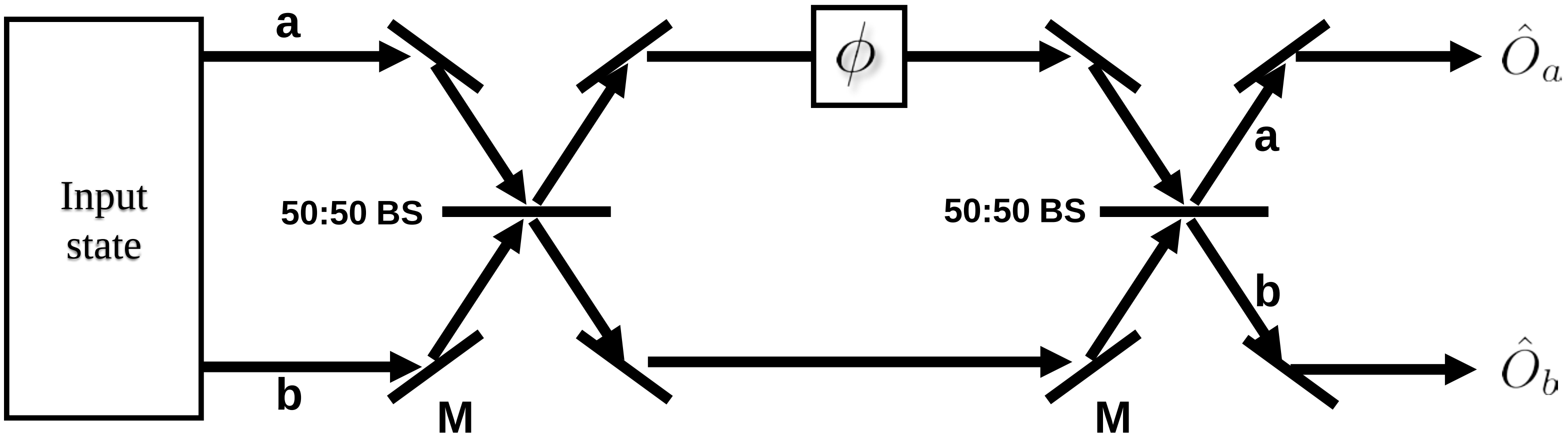
2. Sample loss : $\eta_a < 1, \eta_b = 1$



$$\Delta^2\phi = \frac{\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2}{(\partial \langle \hat{O} \rangle / \partial \phi)^2}$$

PRA **96**, 062304 (2017), C. Oh *et al.*

Double Homodyne Detection



- Homodyne detection

$$\Delta^2\phi = \frac{\langle \hat{O}^2\rangle - \langle \hat{O}\rangle^2}{(\partial\langle \hat{O}\rangle/\partial\phi)^2}$$

- Coherent & Squeezed vacuum

$$\hat{O} = \hat{X}_{\varphi_a} + \hat{X}_{\varphi_b} \qquad \Delta^2\phi = \frac{1}{\alpha^2(e^{2r}+1)} \sim \frac{1}{\bar{n}^2}$$

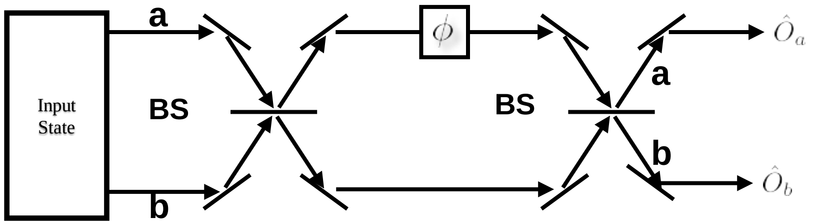
- Two mode squeezed state

$$\hat{O} = \hat{X}_a\hat{X}_b$$

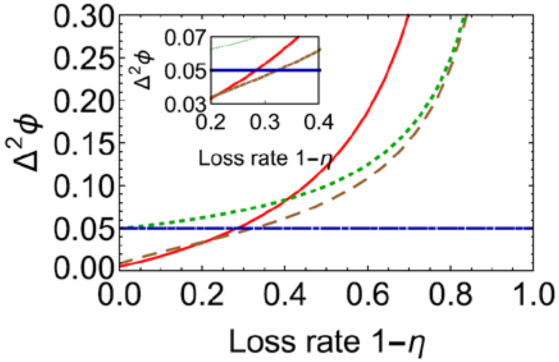
$$\Delta^2\phi = \frac{1}{\sqrt{2+2e^{8r}} - e^{4r} - 1} \xrightarrow{r\gg 1} \frac{\sqrt{2}+1}{4\bar{n}^2}$$

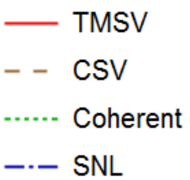
Double Homodyne Detection

- Homodyne detection



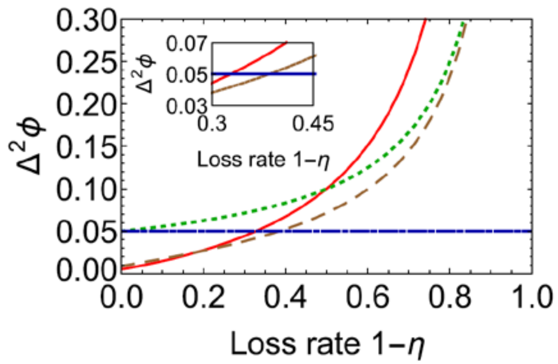
1. Symmetric losses : $\eta_a = \eta_b$





$\bar{n} = 10$

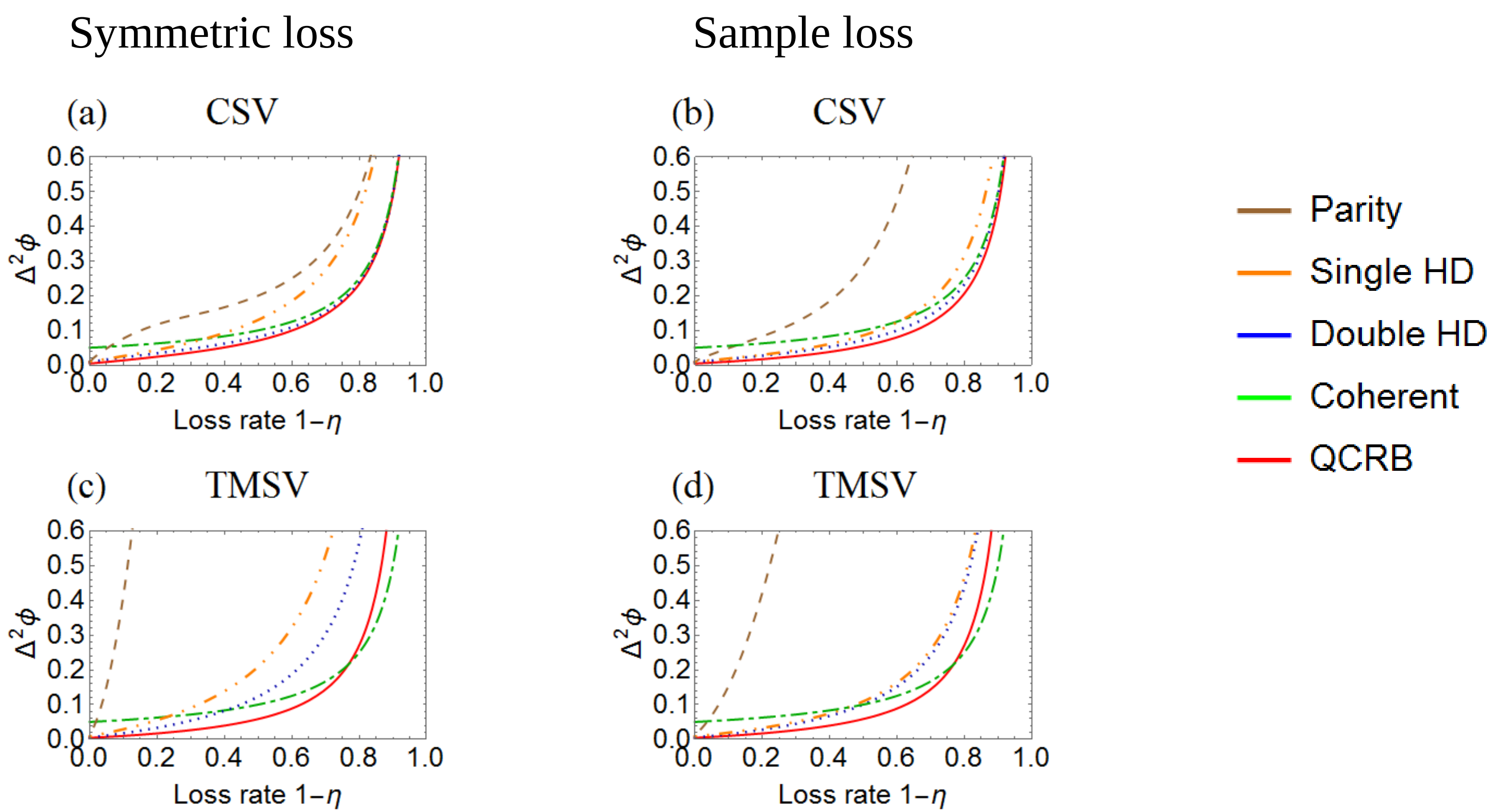
2. Sample loss : $\eta_a < 1, \eta_b = 1$



$$\Delta^2\phi = \frac{\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2}{(\partial \langle \hat{O} \rangle / \partial \phi)^2}$$

PRA **96**, 062304 (2017), C. Oh *et al.*

Comparison of Phase Sensitivity



- Double homodyne detection gives the best phase sensitivity even though it does not saturate QCRB.
- Quantum enhancement is also maintained by using double homodyne detection even under a photon-loss channel.

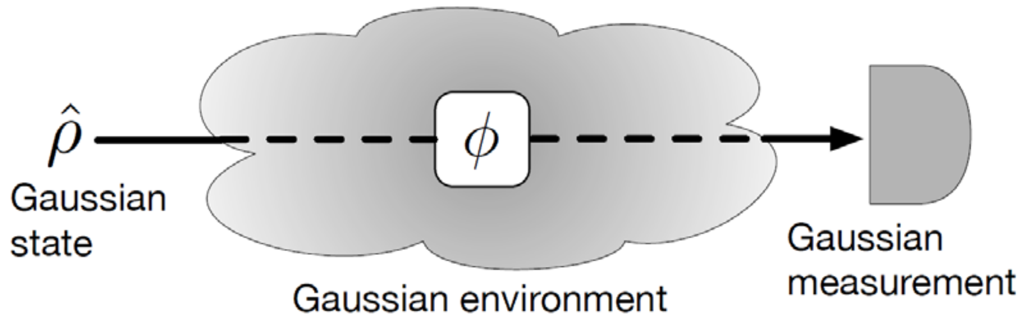
PRA **96**, 062304 (2017), C. Oh *et al.*

Conclusion

- We investigated phase sensitivity in lossy Mach-Zehnder interferometer with practical input states and measurement setups.
- Both TMSV and CSV state maintain quantum enhancement even under photon-loss in a view of quantum Fisher information.
- Parity detection is fragile against photon-loss even when the loss rate is very small.
- Homodyne detection is robust to photon-loss so that it allows quantum enhancement in lossy Mach-Zehnder interferometer.

PRA **96**, 062304 (2017), C. Oh *et al*.

Single-mode Gaussian metrology



$$\hat{\rho} = \hat{D}(\alpha)\hat{S}(\xi)\hat{\rho}_T\hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)$$
$$= \sum_{n=0}^{\infty} p(n)\hat{D}(\alpha)\hat{S}(\xi)|n\rangle\langle n|\hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)$$

$$\alpha = |\alpha|e^{i\theta_c}$$
$$\xi = re^{i\theta_s}$$

$$N_{\rm in} = |\alpha|^2 + \sinh^2 r + (1 + 2\sinh^2 r)n_{\rm th}.$$

Any single-mode Gaussian state can be written in the above form.

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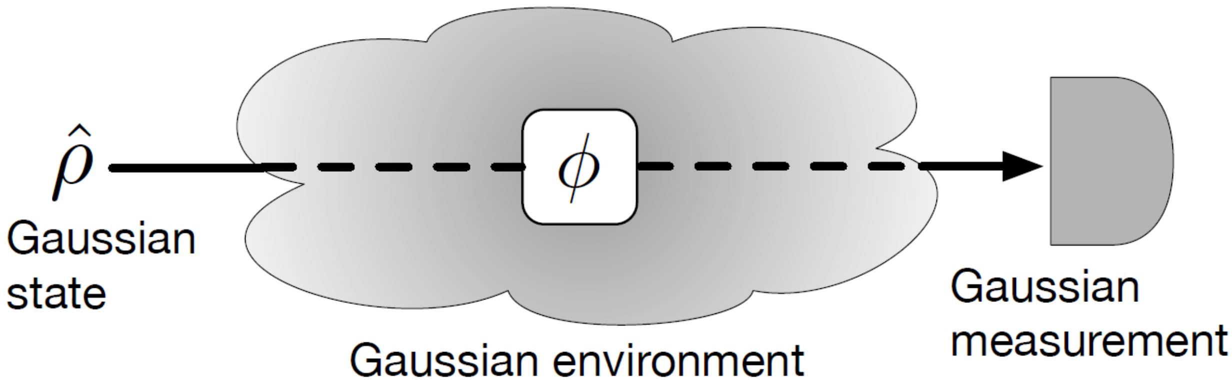
Gaussian noise



$$\frac{d\hat{\rho}(t)}{dt} = \frac{\gamma}{2}\left[n_e\mathcal{L}[\hat{a}^\dagger] + (n_e+1)\mathcal{L}[\hat{a}]\right]\hat{\rho}(t)$$

$$\sigma' = (1-\eta)\sigma_\infty + \eta\sigma \qquad \eta = e^{-\gamma t}$$

$$d' = \sqrt{\eta}d$$

Gaussian state → Gaussian state

$$\hat{\rho} = \hat{D}(\alpha)\hat{S}(\xi)\hat{\rho}_T\hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha) \qquad \hat{\rho}' = \hat{D}(\alpha')\hat{S}(\xi')\hat{\rho}'_T\hat{S}^\dagger(\xi')\hat{D}^\dagger(\alpha')$$

Gaussian measurement

Definition : Gaussian measurement is defined as a measurement scheme that gives Gaussian probabilities of outcomes for all Gaussian states.

Implementation : Gaussian measurement can be implemented by adding ancilla and homodyne detections.

$$\hat{\Pi}_\beta = \frac{1}{\pi}\hat{D}(\beta)\hat{\Pi}^0\hat{D}^\dagger(\beta)$$

where $\hat{\Pi}^0$ is a density operator of a Gaussian state.

$$\Longrightarrow \quad \hat{\Pi}_\beta = \frac{1}{\pi}\hat{D}(\beta)|\zeta\rangle\langle\zeta|\hat{D}^\dagger(\beta)$$

where $|\zeta\rangle$ is the squeezed vacuum state with squeezing parameter ζ.

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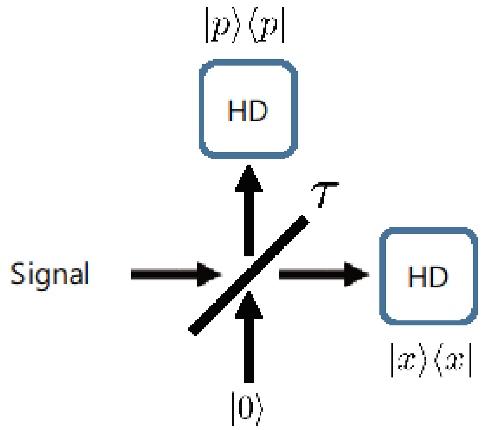
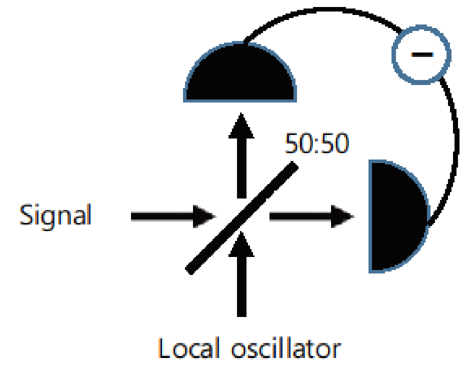
Gaussian measurement

$$\hat{\Pi}_\beta = \frac{1}{\pi}\hat{D}(\beta)|\zeta\rangle\langle\zeta|\hat{D}^\dagger(\beta) \qquad \zeta = se^{i\varphi}$$





$$s = \ln\sqrt{\frac{1-\tau}{\tau}} \qquad \beta = \frac{x}{\sqrt{2\tau}} + i\frac{p}{\sqrt{2(1-\tau)}}$$

$s \to \infty$ $\quad \hat{\Pi}_{x_\varphi} = |x_\phi\rangle\langle x_\phi|$ Homodyne detection

$s = 0$ $\quad \hat{\Pi}_\beta = \frac{1}{\pi}|\beta\rangle\langle\beta|$ Heterodyne detection

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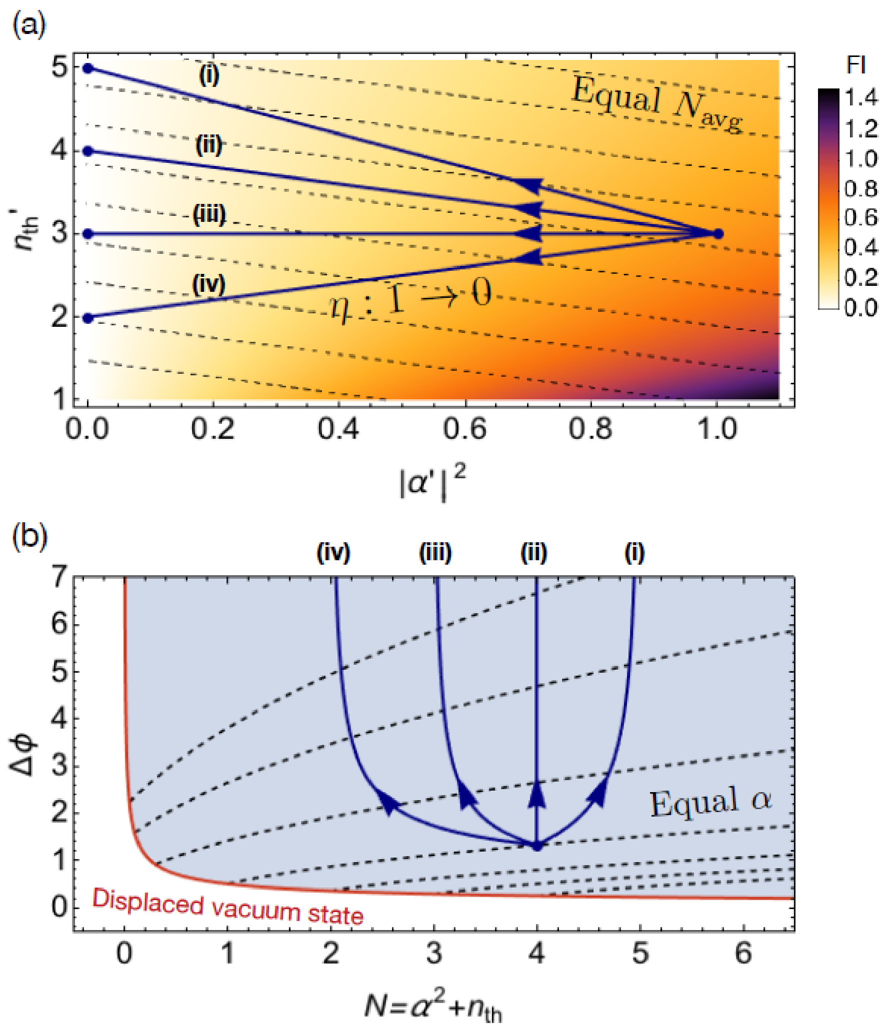
Displaced thermal state

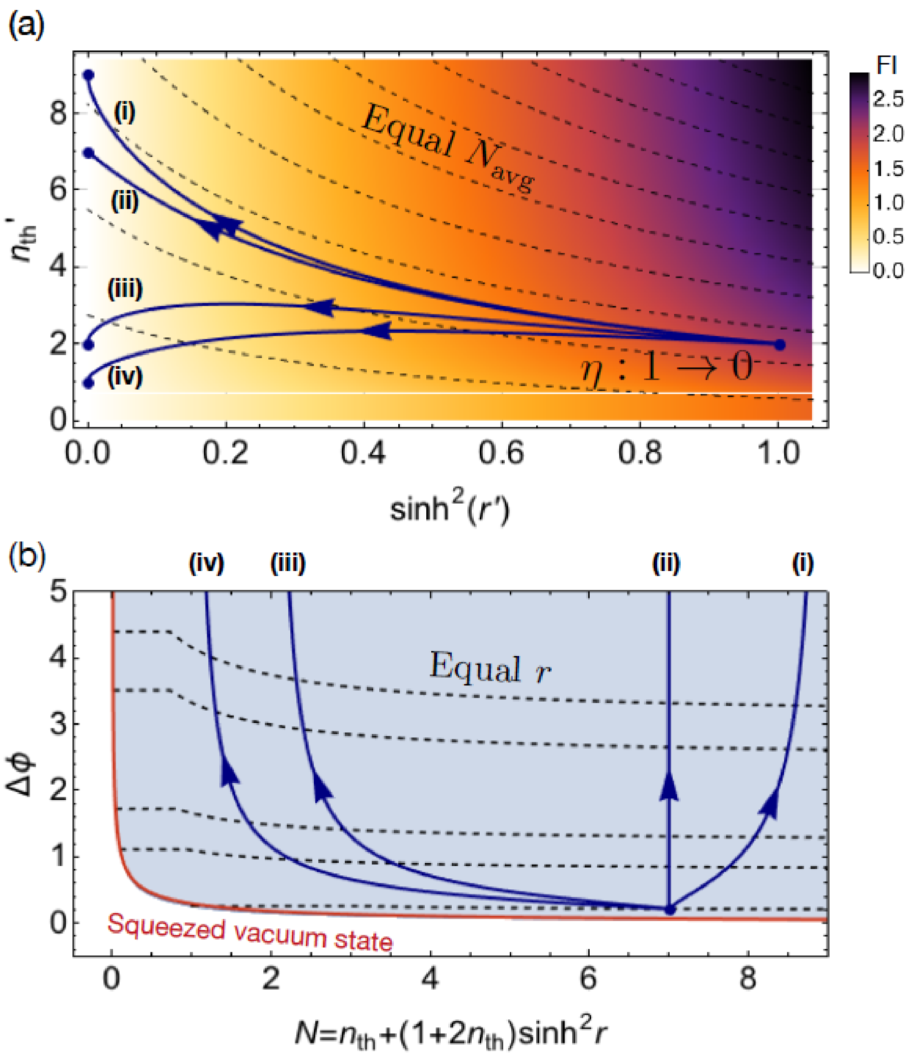


$$\alpha^2 = 1 \quad n_{\mathrm{th}} = 2$$

- Homodyne is always optimal.
- For a fixed displacement, adding thermal photons decreases the sensitivity.
- Thus, the maximal Fisher information is obtained when the thermal photon does not exist.
- The photon loss

$$\text{(i)}\ n_e = N_{\mathrm{in}} + 2$$
$$\text{(ii)}\ n_e = N_{\mathrm{in}}$$
$$\text{(iii)}\ n_e = n_{\mathrm{th}}$$
$$\text{(iv)}\ n_e = n_{\mathrm{th}} - 1$$

Squeezed thermal state



$$\sinh^2 r = 1, n_{th} = 2$$

- For a fixed total mean photon number, the squeezed vacuum state gives the maximum Fisher information.
- For a fixed squeezing parameter, adding thermal photons increases the sensitivity.
- Thus, the maximal Fisher information is obtained when the squeezing parameter and the number of thermal photons are the maximum.

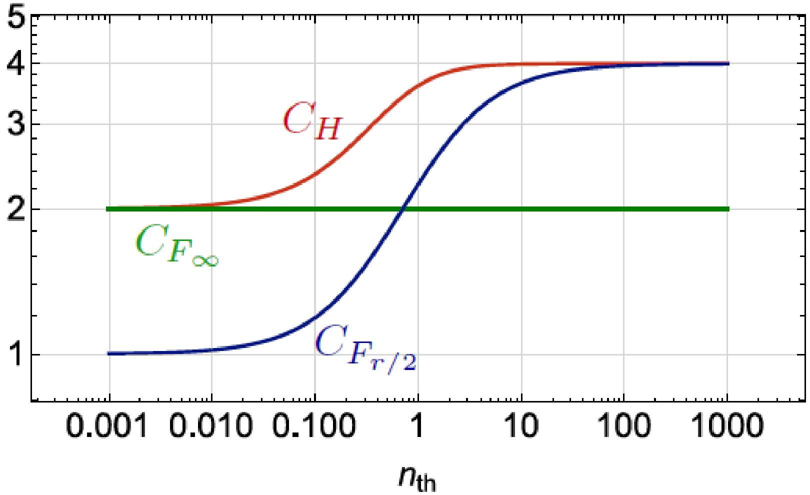
(i) $n_e = N_{\text{in}} + 7$

(ii) $n_e = N_{\text{in}}$

(iii) $n_e = n_{\text{th}}$

(iv) $n_e = n_{\text{th}} - 1$

Squeezed thermal state



$$H = \frac{2(1+2n'_{\rm th})^2 \sinh^2 2r'}{1+2n'_{\rm th}(1+n'_{\rm th})}$$

$$F_{\rm STS,\infty} = C_{F\infty} \sinh^2 2r$$

$$F_{\rm STS,r/2} = C_{Fr/2} \sinh^2 2r$$

- Homodyne detection is no longer optimal for squeezed thermal state.
(It is optimal for squeezed vacuum state.)

- Among Gaussian measurements, homodyne detection is the best for $n_{\rm th} \leq 1/\sqrt{2}$, while a particular Gaussian measurement with $s = r/2$ and $\varphi = 0$ is the best for $n_{\rm th} \geq 1/\sqrt{2}$

- Gaussian measurement is not optimal in general.

- In the limit of small or large $n_{\rm th}$, Gaussian measurement is almost optimal.

Conclusion

- We have investigated Gaussian metrology with general Gaussian measurements.
- Gaussian measurement is optimal for displaced thermal states and squeezed vacuum state.
- Gaussian measurement is not optimal for squeezed thermal states and displaced squeezed thermal states.

THANK YOU FOR YOUR ATTENTION