Resonant fluorescence from a driving two-levels atom

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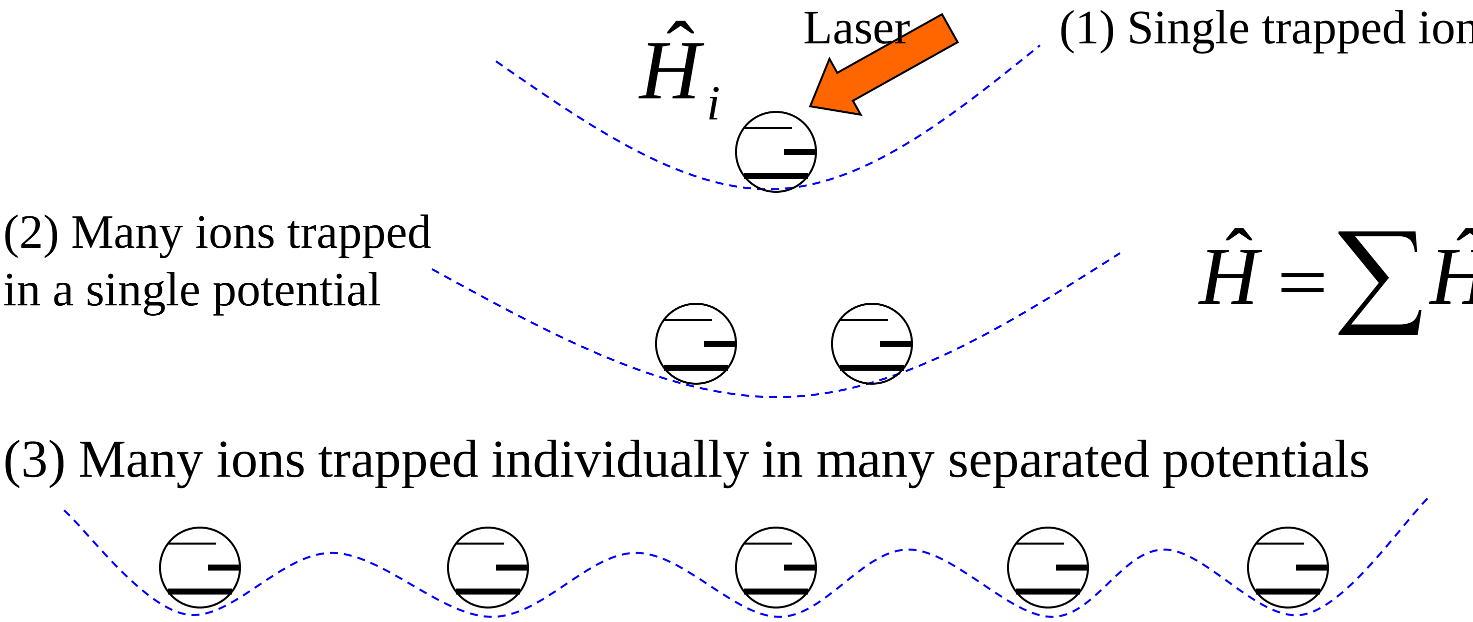
Background 1: trapped ions



$$\hat{H} = \sum \hat{H}_i + V_{ij}$$

Internal atomic motion

External vibration

$$\hat{H}_i = \frac{1}{2}\hbar\omega\hat{\sigma}_z + \hbar\nu(\hat{a}^+\hat{a} + \frac{1}{2}) + \hat{\vec{p}} \cdot \hat{\vec{E}}(x)$$

Interacting with external fields

$$\hat{H}_i = \frac{1}{2}\hbar\omega\hat{\sigma}_z + \hbar\nu(\hat{a}^+\hat{a} + \frac{1}{2}) + \hat{\vec{p}}\cdot\hat{\vec{E}}(x)$$

$$\hat{\vec{E}}^{(+)}(t,x) = \vec{E}_L^{(+)}(x,t) + \hat{\vec{E}}_Q^{(+)}(x,t)$$

Classical field: Laser

Quantum field：cavity QED and the dissipation theory.

Note that:

harmonic oscillation $\hat{x} = \sqrt{\frac{\hbar}{2m\nu}}(\hat{a}^+ + \hat{a})$ $\hat{\vec{p}} = e\vec{\mu}(\hat{\sigma}_- + \hat{\sigma}_+)$ **atomic dipole moment**

Two basic coupling： $\hat{a}\hat{\sigma}_+ + \text{h.c.}$ $\hat{c}\hat{\sigma}_+ + \text{h.c.}$

together with Coulomb interaction within the many-ion system, $\hat{H} = \sum \hat{H}_i + V_{ij}(\hat{x}_i - \hat{x}_j)$

allow us to construct many types of controllable Hamiltonian for realizing

Quantum Information Processing.

M. Zhang, and L. F. Wei, PHYSICAL REVIEW A 83, 064301 (2011)

L. F. Wei, M. Zhang, H. Y. Jia, and Y. Zhao, PHYSICAL REVIEW A 78, 014306 (2008)

Background 2: fundamentals of Quantum Optics

2.1. Field Quantization (very like to the classical field, beside of the dimensionless operator)

$$\hat{\vec{E}} = \hat{\vec{E}}^{(+)} + \hat{\vec{E}}^{(-)} \quad \text{with} \quad [\hat{E}^{(-)}(r)]^{+} = \hat{E}^{(+)}(r)$$

$$\hat{\vec{E}}^{(+)}(t,r) = i\sum_{k,s} \vec{E}_{k,s}\hat{a}_{k,s}e^{-i(\omega_k t - \vec{k}\cdot\vec{r})} \quad \text{with} \quad \vec{E}_{k,s} = \vec{e}_s\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}$$

and the so-called creation and destroy operators. $[\hat{a}_{k,s}, \hat{a}^{+}_{k',s'}] = \delta_{k,k'}\delta_{s,s'}$

Note that, in Heisenberg picture,

(1) The time-dependent field operator satisfies Maxwell's equations, i.e.,

$$\nabla^2\hat{\vec{E}} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\hat{\vec{E}} = 0$$

(2) Cavity' field, standing wave, single-frequency, includes:

$$\hat{\vec{E}}_{+k} = i\vec{E}_k\hat{a}_k e^{-i(\omega_k t - \vec{k}\cdot\vec{r})} - i\vec{E}_k\hat{a}^{+}_k e^{i(\omega_k t - \vec{k}\cdot\vec{r})}$$ Right traveling wave

$$\hat{\vec{E}}_{-k} = -i\vec{E}_k\hat{a}_k e^{-i(\omega_k t + \vec{k}\cdot\vec{r})} + i\vec{E}_k\hat{a}^{+}_k e^{i(\omega_k t + \vec{k}\cdot\vec{r})}$$ Left traveling, with **half-wave loss**

$$\hat{\vec{E}}_{+k} + \hat{\vec{E}}_{-k} = (\hat{a}_k e^{-i\omega_k t} + \hat{a}^{+}_k e^{i\omega_k t})2\sin(\vec{k}\cdot\vec{x})$$

The well-known field in cavity, with Heisenberg picture.

2.2. Atom-field interaction

$$\hat{V} = -e\hat{\vec{x}} \cdot \hat{\vec{E}}$$ Electric dipole interaction energy

$$\hat{\vec{E}}(t,\vec{r}) = i\sum_{k,s} \vec{E}_{k,s}\hat{a}_{k,s}e^{i\vec{k}\cdot\vec{r}} + \text{h.c} \qquad \vec{E}_{k,s} = \vec{e}_s\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}$$

$$\hat{\vec{x}} = \hat{1}\hat{\vec{x}}\hat{1} = (|g\rangle\langle g| + |e\rangle\langle e|)\hat{\vec{x}}(|g\rangle\langle g| + |e\rangle\langle e|) = \vec{\mu}(\hat{\sigma}_+ + \hat{\sigma}_-)$$

$$\vec{\mu} = \langle g|\vec{x}|e\rangle = \langle e|\vec{x}|g\rangle \quad \langle g|\vec{x}|g\rangle = \langle e|\vec{x}|e\rangle = 0 \quad \hat{\sigma}_+ = |e\rangle\langle g| \quad \hat{\sigma}_- = |g\rangle\langle e|$$

$$\hat{V} = i\hbar\sum_{k,s}(\hat{\sigma}_+ + \hat{\sigma}_-)(g_{k,s}\hat{a}_{k,s} - g_k^*\hat{a}_{k,s}^+) \approx i\hbar\sum_{k,s}(g_{k,s}\hat{\sigma}_+\hat{a}_{k,s} - g_k^*\hat{\sigma}_-\hat{a}_{k,s}^+)$$

$$g_{k,s} = -e(\vec{\mu}\cdot\vec{e}_s)e^{i\vec{k}\cdot\vec{r}_a}\sqrt{\frac{\omega_k}{2\varepsilon_0\hbar V}}$$

Rotating wave approximation, weak coupling. $g_{k,s} << \omega_a$

Atomic position $\vec{r}_a$ **may be useful for the position-controllable trapped ions**

2.3. JC Model

The total Hamiltonian for the atom-field system is thus,

$$\hat{H} = \hat{H}_{\mathrm{f}} + \hat{H}_{\mathrm{a}} + \hat{V}$$

with

$$\hat{H}_{\mathrm{f}} = \sum_{k,s} \hbar\omega_{k,s}(\hat{a}^{+}_{k,s}\hat{a}_{k,s} + 1/2)$$

$$\hat{H}_{\mathrm{a}} = E_g|g\rangle\langle g| + E_e|e\rangle\langle e| = \frac{1}{2}\omega_a\hat{\sigma}_z + \mathrm{constant}$$

$$\omega_a = (E_e - E_g)/\hbar \qquad \hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|$$

Under the rotating wave approximation, **JC model:**

$$\hat{H} = \frac{1}{2}\hbar\omega_a\hat{\sigma}_z + \sum_{k,s}\hbar\omega_{k,s}(\hat{a}^{+}_{k,s}\hat{a}_{k,s} + 1/2) + i\hbar\sum_{k,s}(g_{k,s}\hat{\sigma}_{+}\hat{a}_{k,s} - g^{*}_{k,s}\hat{\sigma}_{-}\hat{a}^{+}_{k,s})$$

This Hamiltonian implies that atom is entangling with the field.

2.4. The relation between field and atom

$$\hat{H}=\frac{1}{2}\hbar\omega_a\hat{\sigma}_z+\sum_{k,s}\hbar\omega_{k,s}(\hat{a}^+_{k,s}\hat{a}_{k,s}+1/2)+i\hbar\sum_{k,s}(g_{k,s}\hat{\sigma}_+\hat{a}_{k,s}-g^*_{k,s}\hat{\sigma}_-\hat{a}^+_{k,s})$$

Based on the above Hamiltonian, scientists get two solutions.

(1) Field:

$$\hat{\vec{E}}^{(+)}(t,\vec{r})=\hat{\vec{E}}^+_0(t,\vec{r})-i\sum_{k,s}\vec{E}_{k,s}g^*_{k,s}e^{i\vec{k}\cdot\vec{r}_a}\int_0^t dt'\hat{\sigma}_-(t')e^{i\omega_k(t'-t)}$$

(without atom) **(atom dependent)**

(2) Master equation for atom:

$$\frac{d\hat{\rho}}{dt}=\frac{i}{\hbar}[\hat{H}_I,\hat{\rho}]+\frac{\Gamma}{2}(2\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+-\hat{\sigma}_+\hat{\sigma}_-\hat{\rho}-\hat{\rho}\hat{\sigma}_+\hat{\sigma}_-)$$

→ Decay

$$+\frac{\gamma}{2}(2\langle e|\hat{\rho}|e\rangle|e\rangle\langle e|-\hat{\rho}|e\rangle\langle e|-|e\rangle\langle e|\hat{\rho})$$

→ Dephasing

M. Oraszag,
“Quantum optics including noise Reduction, trapped ions, quantum trajectories, and decoherence”,
Springer-Verlag Berlin Heidelberg, 2000.

The form of point dipole emission:

$$\hat{\vec{E}}^{(+)}(t,\vec{r}) = \hat{\vec{E}}_0^{+}(t,\vec{r}) - \frac{\omega_a^2}{4\pi c^2 r^3}(\vec{\mu}\times\vec{r}\times\vec{r})\hat{\sigma}_-(t-r/c)$$

"The result is, of course, the familiar retarded field generated by a point dipole plus a freely propagating component"

\sigma will be solved by using the master equation (Approximate solution for the JC model)

SECOND SERIES, VOL. 188, NO. 5 25 DECEMBER 1969

Physical review

Power Spectrum of Light Scattered by Two-Level Systems **Eq. (2.11)**

B. R. MOLLOW*

National Aeronautics and Space Administration, Electronics Research Center, Cambridge, Massachusetts

(Received 2 September 1969)

PHYSICAL REVIEW A VOLUME 13, NUMBER 6 JUNE 1976

Theory of resonance fluorescence* **Eq. (16)**

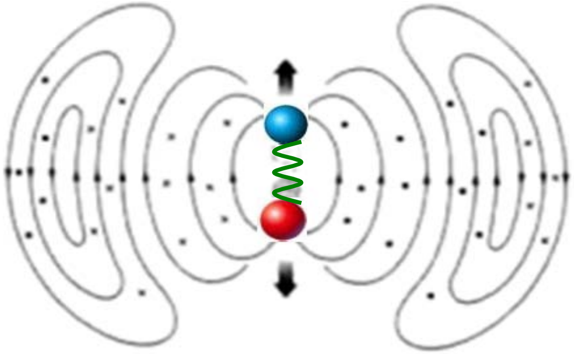
H. J. Kimble and L. Mandel

Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

(Received 23 December 1975)

3. Fluorescence

Master equation+ Maxwell‘s equations



3.1 Classical dipole radiation

$$\vec{p}(t) = \vec{p}_0 e^{-i\omega t} + \text{c.c.}$$

Using the retarded potential formulation,
and limiting within the far-field regime: (see, text book)

$$\vec{E} = \frac{\omega^2}{4\pi\varepsilon_0 c^2}\left\{\frac{\vec{p}(t-r/c)}{r} - \frac{\vec{e}_r[\vec{e}_r \cdot \vec{p}(t-r/c)]}{r}\right\} + \text{c.c.}$$

If the dipole-vibration is quantum mechanical, its emission is also quantum mechanical, otherwise, the Maxwell’s equation does not hold.

$$\vec{p}(t) \rightarrow \hat{\vec{p}}(t) \qquad \vec{E}(t) \rightarrow \hat{\vec{E}}(t)$$

We consequently compute the dipole-vibration, by using atomic master equation.

3.2. Master equation for a two-levels atom

The Hamiltonian for two-levels atom driving by a classical light.

$$\hat{H}_I = \frac{1}{2}\hbar\delta\hat{\sigma}_z + \hbar\Omega(e^{i\phi}\hat{\sigma}_- + e^{-i\phi}\hat{\sigma}_+)$$ **(interacting picture)**

$$\delta = \omega_a - \omega_l$$ **(considering a small detuning in general)**

Master equation

$$\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar}[\hat{H}_I, \hat{\rho}] + \frac{\Gamma}{2}(2\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \hat{\sigma}_+\hat{\sigma}_-\hat{\rho} - \hat{\rho}\hat{\sigma}_+\hat{\sigma}_-)$$

→ Decay

$$+\frac{\gamma}{2}(2\langle e|\hat{\rho}|e\rangle|e\rangle\langle e| - \hat{\rho}|e\rangle\langle e| - |e\rangle\langle e|\hat{\rho})$$

→ Dephasing

3. 3. Density operator

The time-dependent Density operator for the two-levels system

$$\hat{\rho} = |\psi'\rangle\langle\psi'| = \rho_{gg}|g\rangle\langle g| + \rho_{ee}|e\rangle\langle e| + \rho_{eg}|e\rangle\langle g| + \rho_{ge}|g\rangle\langle e|$$

At any time, the state can be formally written as

$|\psi'\rangle = \alpha|g\rangle + \beta|e\rangle$ **with two complex numbers** $\alpha = a + ib \quad \beta = c + id$

Obviously,

$$\rho_{gg} = \alpha^* \alpha = a^2 + b^2$$

$$\rho_{ee} = \beta^* \beta = c^2 + d^2$$

$$\rho_{eg} = \rho_{ge}^* = \alpha^* \beta = (a - ib)(c + id) = ac + bd + i(ad - bc)$$

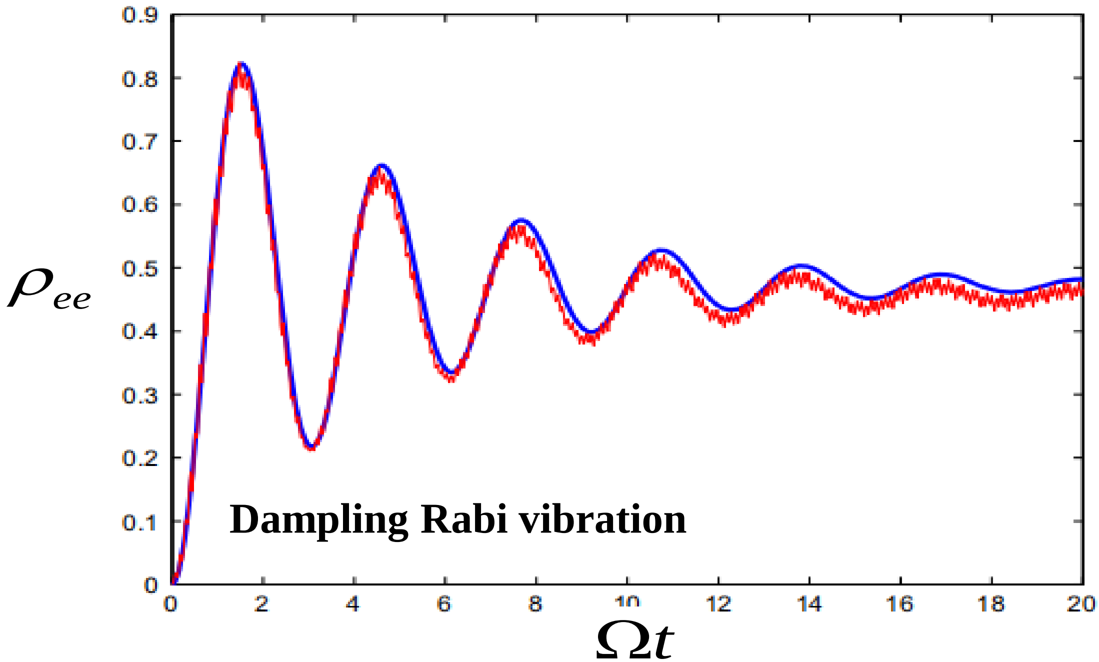
$$\rho_{gg} + \rho_{ee} = a^2 + b^2 + c^2 + d^2 = 1$$ **(normalization)**

Thus, four real numbers a, b, c, d can be solved by above four equations, with the Density matrix elements.

3. 4. Numerical solution for the density matrix elements

$$\frac{d\rho_{ee}}{dt} = i\Omega(\rho_{ge}e^{-i\phi} - \rho_{eg}e^{i\phi}) - \Gamma\rho_{ee} \qquad \rho_{gg} + \rho_{ee} = 1$$

$$\frac{d\rho_{eg}}{dt} = i(\rho_{gg}\Omega e^{-i\phi} - \rho_{ee}\Omega e^{-i\phi} - \delta\rho_{eg}) - \frac{\Gamma+\gamma}{2}\rho_{eg} \qquad \rho_{ge} = \rho_{eg}^{*}$$

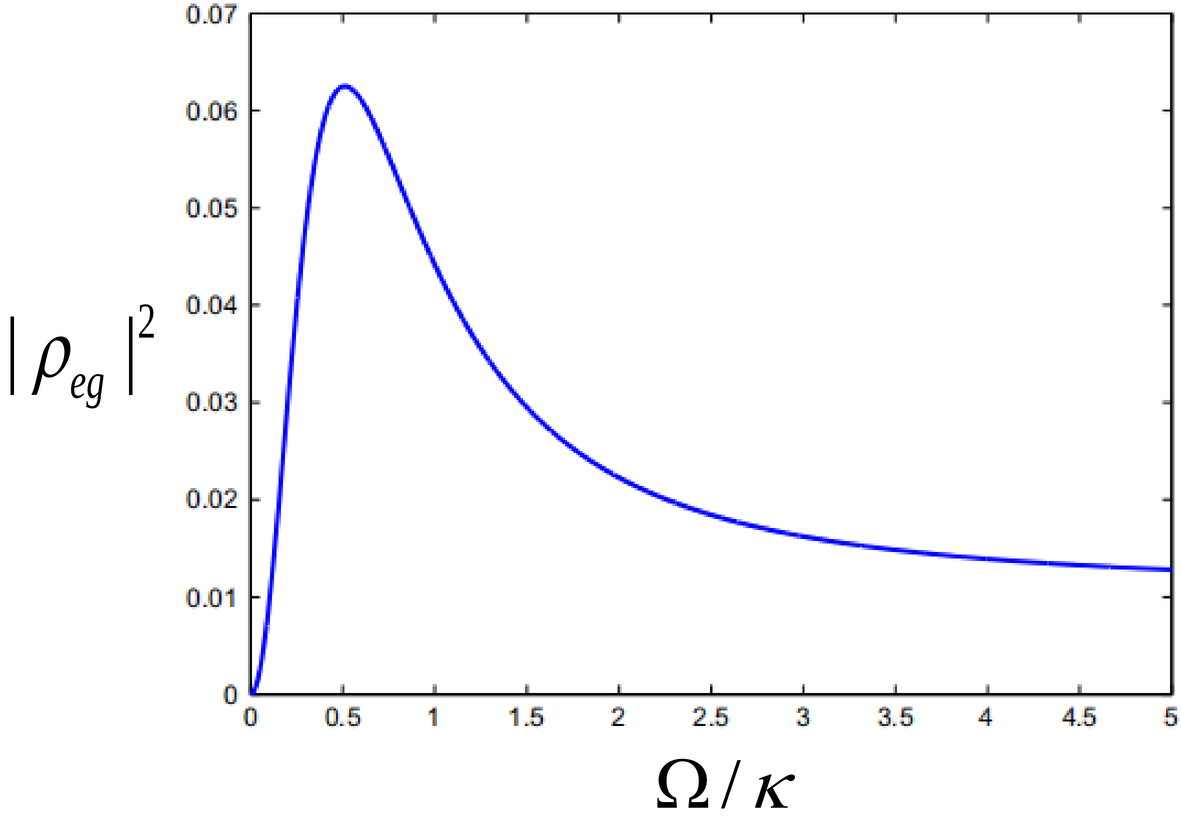


M. Zhang, W.Z. Jia, L.F. Wei, Physica B 432 (2014)

3. 5. Steady-state solution

$$\dot{\rho}_{eg} = \dot{\rho}_{ee} = 0$$

$$\rho_{eg} = \frac{\Omega e^{-i\phi}(1-2\rho_{22})(i\kappa+\delta)}{\kappa^2+\delta^2} \qquad \rho_{ee} = \frac{2\Omega^2\kappa}{\Gamma(\kappa^2+\delta^2)+4\Omega^2\kappa}$$



writing

$$\kappa = (\Gamma+\gamma)/2$$

for short.

3.4 The time-dependent positional operator

$$\langle \psi | \vec{x} | \psi \rangle = \langle \psi' | e^{i\hat{H}_0 t} \vec{x} e^{-i\hat{H}_0 t} | \psi' \rangle$$

$$= \langle \psi' | e^{i\hat{H}_0 t} (| g \rangle\langle g | + | e \rangle\langle e |) \vec{x} (| g \rangle\langle g | + | e \rangle\langle e |) e^{-i\hat{H}_0 t} | \psi \rangle$$

$$= \vec{\mu} \langle \psi' | (e^{-i\omega t} | g \rangle\langle e | + e^{i\omega t} | e \rangle\langle g |) | \psi \rangle$$

$$= \langle i | (e^{-i\omega t} \hat{\sigma}_-(t) + e^{i\omega t} \hat{\sigma}_+(t)) | i \rangle$$

with transition matrix elements:

$$\vec{\mu} = \langle g | \vec{x} | e \rangle = \langle e | \vec{x} | g \rangle \qquad \langle g | \vec{x} | g \rangle = \langle e | \vec{x} | e \rangle = 0$$

and

$$\hat{\sigma}_-(t) = \frac{1}{\langle i | \hat{\rho}(t) | i \rangle} \hat{\rho}(t) | g \rangle\langle e | \hat{\rho}(t) = \frac{\rho_{eg}}{\langle i | \hat{\rho}(t) | i \rangle} \hat{\rho}(t)$$

$$\hat{\sigma}_-(t) = [\hat{\sigma}_-(t)]^+ \qquad \hat{\rho}(t) = | \psi' \rangle\langle \psi' |$$

3.5 Field Quantization

$$\hat{\vec{x}}^{(+)}(t) = \vec{\mu} e^{-i\omega t} \hat{\sigma}_-(t) \qquad \hat{\vec{x}}^{(-)}(t) = [\hat{\vec{x}}^{(+)}(t)]^+$$

The positive frequency of dipole moment is thus

$$\hat{\vec{p}}^+(t) = -e\hat{\vec{x}}^{(-)}(t) = \vec{p}_0 e^{i\omega t} \hat{\sigma}_-(t)$$

Replacing the classical dipole in retarded field:

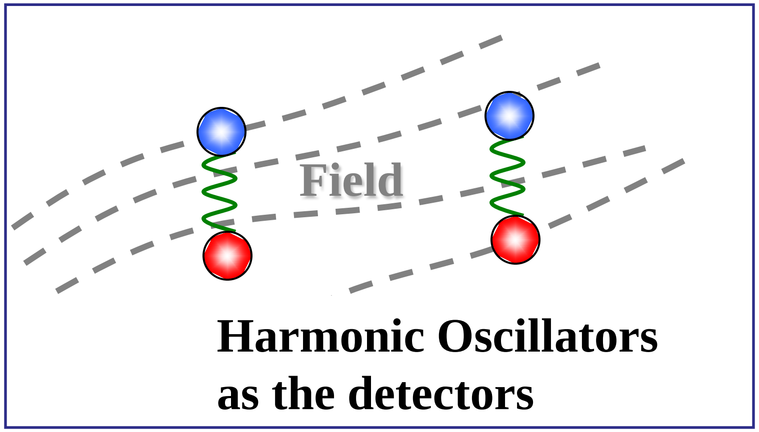
$$\hat{\vec{E}}^{(+)} = \frac{\omega^2}{4\pi\varepsilon_0 c^2} \left\{ \frac{\hat{\vec{p}}(t - r/c)}{r} - \frac{\vec{e}_r[\vec{e}_r \cdot \hat{\vec{p}}(t - r/c)]}{r} \right\}$$

$$= \vec{E}_0^{(+)}(r) e^{-i\omega t} \hat{\sigma}_-(t - r/c)$$

with an amplitude:

$$\vec{E}_0^{(+)}(r) = \frac{\omega^2}{4\pi\varepsilon_0 c^2} [\frac{\vec{p}_0}{r} - \frac{\vec{e}_r(\vec{e}_r \cdot \hat{\vec{p}}_0)}{r}] e^{ikr}$$

4. Detectors



(1) The difference between classical and quantum fields is on operators. We should to read out the characters of operators.
(2) Using Heisenberg equation to compute the motion of detector, because the it is most same to the classical ones (Ehrenfest theorem).

4.1 Two detectors in the classical fields

$$\vec{E}_c = \vec{E}_0(r)e^{i\vec{k}\cdot\vec{r}}e^{-i\omega t} + \vec{E}_0(r)e^{-i\vec{k}\cdot\vec{r}}e^{i\omega t}$$

Under resonant approximation,

$$\hat{H} = \hbar g(r_1)\hat{a}^+ + \hbar g^*(r_1)\hat{a} + \hbar g(r_2)\hat{b}^+ + \hbar g^*(r_2)\hat{b}$$

$$[\hat{a},\hat{b}] = [\hat{a}^+,\hat{b}] = [\hat{a},\hat{b}^+] = [\hat{a}^+,\hat{b}^+] = 0$$

Classical field does not generate correlation between two detectors.

Heisenberg equation

$$i\hbar \frac{d}{dt}\hat{U}(t)\left|\psi_0\right\rangle = \hat{H}(t)\hat{U}(t)\left|\psi_0\right\rangle$$ → Schrödinger Eq.

$\hat{U}(t)$ → Evolution operator

$$\frac{d\hat{U}}{dt} = \frac{-i}{\hbar}\hat{H}\hat{U} \qquad \frac{d\hat{U}^+}{dt} = \frac{i}{\hbar}\hat{U}^+\hat{H}$$

$$\frac{d}{dt}(\hat{U}^+\hat{x}\hat{U}) = \frac{i}{\hbar}\hat{U}^+\hat{H}\hat{x}\hat{U} - \frac{i}{\hbar}\hat{U}^+\hat{x}\hat{H}\hat{U} = \frac{i}{\hbar}\hat{U}^+[\hat{H},\hat{x}]\hat{U}$$ → Heisenberg Eq.

$$\hat{H} = \hbar g(r_1)\hat{a}^+ + \hbar g^*(r_1)\hat{a} + \hbar g(r_2)\hat{b}^+ + \hbar g^*(r_2)\hat{b}$$

Dynamical equation of oscillators:

$$\frac{d}{dt}\hat{x}_1(t) = \frac{i}{\hbar}\hat{U}^+[\hat{H},\hat{x}_1(0)]\hat{U} = 2x_0 \operatorname{Im} g(r_1)$$ with $\hat{x}_1(0) = x_0(\hat{a}^+ + \hat{a})$

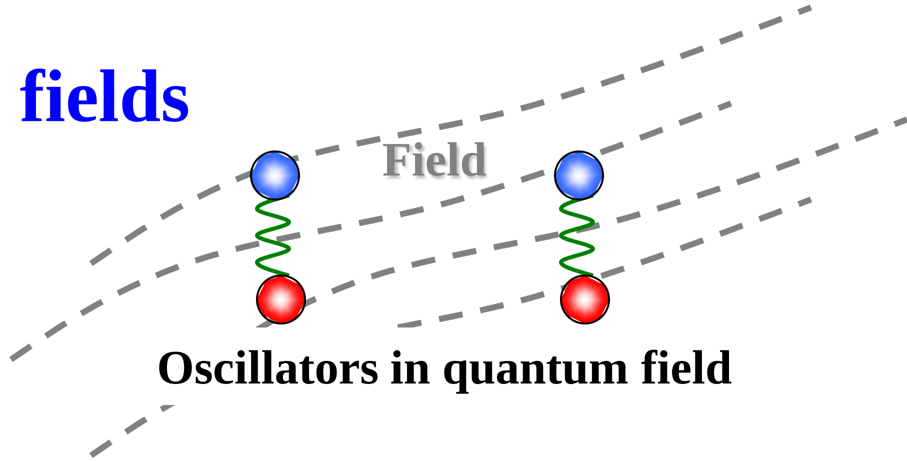
and where $[\hat{H},\hat{x}_1(0)] = x_0\hbar g^*(r_1) - x_0\hbar g(r_1)$

Two detectors (harmonic oscillators) are independent.

4.2 Detectors within quantum fields

$$\hat{\vec{E}}^{(+)}(r,t) = \vec{E}_0^{(+)}(r)e^{-i\omega t}\hat{\sigma}_-(t - r/c)$$

$$\hat{\vec{E}}^{(-)}(r,t) = \vec{E}_0^{(-)}(r)e^{i\omega t}\hat{\sigma}_+(t - r/c)$$



Oscillators in quantum field

In the resonant approximation,

$$\hat{H} = \hbar g(r_1)\hat{\sigma}_-(t_1)\hat{a}^+ + \hbar g^*(r_1)\hat{\sigma}_+(t_1)\hat{a}$$

$$+ \hbar g(r_2)\hat{\sigma}_-(t_2)\hat{b}^+ + \hbar g^*(r_2)\hat{\sigma}_+(t_2)\hat{b}$$

$$t_1 = t - r_1/c \qquad t_2 = t - r_2/c$$

Solving again Heisenberg equation for detector 1:

$$\frac{d}{dt}\hat{x}_1(t) = \frac{i}{\hbar}\hat{U}^+[\hat{H}, \hat{x}_1(0)]\hat{U} \qquad \hat{x}_1(0) = x_0(\hat{a}^+ + \hat{a})$$

Heisenberg equation

$$\frac{d}{dt}\hat{x}_1(t) = x_0\hat{U}^+[ig^*(r_1)\hat{\sigma}_-(t_1) - ig(r_1)\hat{\sigma}_+(t_1)]\hat{U}$$

Within the short time limit, the evolution operator $\hat{U} \approx 1 + \frac{-i}{\hbar}\int_0^t \hat{H}dt$

$$\frac{d\hat{x}_1(t)}{dt} = \hat{v}_1 - \frac{1}{\hbar}\int_0^t dt[\hat{H}_2, g^*(r_1)\hat{\sigma}_-(t_1) - g(r_1)\hat{\sigma}_+(t_1)]$$

Commutator

with detector 2

$$\hat{H}_2 = \hbar g(r_2)\hat{\sigma}_-(t_2)\hat{b}^+ + \hbar g^*(r_2)\hat{\sigma}_+(t_2)\hat{b}$$

$$\hat{\sigma}_-(t_j) = \frac{\rho_{eg}(t_j)}{\langle i|\hat{\rho}(t_j)|i\rangle}\hat{\rho}(t_j) \qquad \hat{\sigma}_+(t_j) = \frac{\rho_{ge}(t_j)}{\langle i|\hat{\rho}(t_j)|i\rangle}\hat{\rho}(t_j)$$

Two detectors can be correlative, becasue

$$[\hat{\rho}(t_j), \hat{\rho}(t_k)] = |\psi_j\rangle\langle\psi_j|\psi_k\rangle\langle\psi_k| - |\psi_k\rangle\langle\psi_k|\psi_j\rangle\langle\psi_j| \neq 0$$

5. Conclusion

(i) Showing the principles for quatizing radiation from a dipole vibration: the master equation for atom, and the Maxwell's equation for field-spreading.

(ii) The key difference between classical field and quantum field are commutation:

$$[\hat{\vec{E}}^{(+)}, \hat{\vec{E}}^{(-)}] \neq 0$$

(iii) Present an explanation to detector.

(iv) Outlooking: applying the result to trapped ions, with quatized central-position of atom:

$$\hat{\vec{E}}^{(+)} = \vec{E}_0^{(+)}(r) e^{-i\omega t} \hat{\sigma}_-(t - r/c)$$

$$\vec{r} \to \hat{\vec{r}} = \sqrt{\frac{\hbar}{2m\nu}} (\hat{a}^+ + \hat{a})$$

$$\hat{\sigma}_-(t_j) = \frac{\rho_{eg}(t_j)}{\langle i | \hat{\rho}(t_j) | i \rangle} \hat{\rho}(t_j)$$

Thanks