Relational model of data over domains with similarities

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Outline

Fuzzy extension of the relational model

[Abiteboul, 2005]

"Traditional DBMSs were applied to business data processing, which typically focused on numbers and character strings. ... When one leaves business data processing, essentially all data is uncertain or imprecise"

- The authors asked for a way to store imprecise data
- but also a way to express imprecise queries and get imprecise answers.

Fuzzy extension of the relational model

		Fuzzyness on data		
		Classical data table	Table of fuzzy sets	Ranked data table
Fuzzyness on functional dependencies	Functional dependency	[Codd, 1970] [Armstrong, 1974]		
	F.D. over domains with similarities			
	Graded F.D. over dom. with similarities			
	Fuzzy functional dependency			

Executable logic

Outline

Relational model [Codd, 1970]

Given:

- a non-empty finite set of attributes Y and
- a family of domains $\{D_y \mid y \in Y\}$,

a database is a relation $\mathcal{R} \subseteq \prod_{y \in Y} D_y$ usually represented in a table

	y_1	y_2	$\ldots$	y_n
$\vdots$	$\vdots$	$\vdots$		$\vdots$
t	$t[y_1]$	$t[y_2]$	$\ldots$	$t[y_n]$
$\vdots$	$\vdots$	$\vdots$		$\vdots$

name	hair	skin	age	eyes	stature
John	black	dark	34	brown	180
Albert	brown	light	32	blue	160
Mary	auburn	lig-int	26	blue	178
Dave	red	light	29	blue	181
Noa	white	dark	32	green	197

Functional dependency [Armstrong, 1974]

`job,experience→salary` "Same job and experience imply same salary"

$\mathcal{R}$ satisfies the functional dependency $A \to B$ if, for all $t_1, t_2 \in \mathcal{R}$,

$$t_1[A] = t_2[A] \text{ implies } t_1[B] = t_2[B].$$

		A				B			
	$\dots$	y_{i_1}	$\dots$	y_{i_n}	$\dots$	y_{j_1}	$\dots$	y_{j_m}	$\dots$
$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$	
t_1	$\dots$	$t_1[y_{i_1}]$	$\dots$	$t_1[y_{i_n}]$	$\dots$	$t_1[y_{j_1}]$	$\dots$	$t_1[y_{j_m}]$	$\dots$
$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$	
t_2	$\dots$	$t_2[y_{i_1}]$	$\dots$	$t_2[y_{i_n}]$	$\dots$	$t_2[y_{j_1}]$	$\dots$	$t_2[y_{j_m}]$	$\dots$
$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$	

Functional Dependencies and Artificial Intelligence

- Logic Programing [Mendelzon,1985]
- Functional Programming [Jones, 2000]
- Specification [Cadoli and Mancini, 2004]
- Neural Networks [Stanikovic and Milovanovic, 2005]
- Grid resource management [Tran and Choi, 2006]
- Software Engineering [Kryszkiewicz and Lasek, 2007]
- Formal Concept Lattices [Belohlavek and Vychodil, 2008]

Idea

Title	Author	Filiation	Conference	Place	Date

Title, Author $\longrightarrow$ Conference; Author $\longrightarrow$ Filiation; Conference $\longrightarrow$ Place,Date

Title	Author	Conference

Author	Filiation

Conference	Place	Date

Title, Author $\longrightarrow$ Title,Author,Filiation,Conference,Place,Date

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Author	Filiation

Conference	Place	Date

Title, Author $\longrightarrow$ Title, Author, Filiation, Conference, Place, Date

Armstrong's axioms

- The language: $\mathcal{L} = \{A \to B \mid A, B \subseteq Y\}$.
- Theory of models ($\models$):

$$\mathcal{R} \models A \to B, \quad \mathcal{R} \models T, \quad T \models A \to B.$$

- Axiomatic system ($\vdash$):
  - Axioms: for all $B \subseteq A$, $\vdash A \to B$.
  - Augmentation rule: $A \to B \vdash AC \to BC$.
  - Transitivity rule: $A \to B, B \to C \vdash A \to C$.
- Soundness and completeness:

$$T \models A \to B \text{ if and only if } T \vdash A \to B$$

- Automatic Reasoning:

What about automated deduction systems?

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What about automated deduction systems?

Simplification Logic $\mathrm{SL_{FD}}$ [Mora et al., 2004, Mora et al., 2006]

The following axiomatic system is equivalent to Armstrong's Axioms:

- **Axioms:** for all $B \subseteq A$, $\vdash A \rightarrow B$
- **Decomposition**: If $C \subseteq B$, $A \rightarrow B \vdash A \rightarrow C$
- **Composition**: $A \rightarrow B,\ C \rightarrow D \vdash AC \rightarrow BD$
- **Simplification**: If $A \cap B = \varnothing$ and $A \subseteq C$, $A \rightarrow B,\ C \rightarrow D \vdash C \smallsetminus B \rightarrow D \smallsetminus B$

Proposition

The following equivalences hold:

- *Decomposition:* $\{A \rightarrow B\} \equiv \{A \rightarrow B \smallsetminus A\}$
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Theorem

$$T \vdash A \rightarrow B \quad \textit{if and only if} \quad T \cup \{\varnothing \rightarrow A\} \vdash \varnothing \rightarrow B$$

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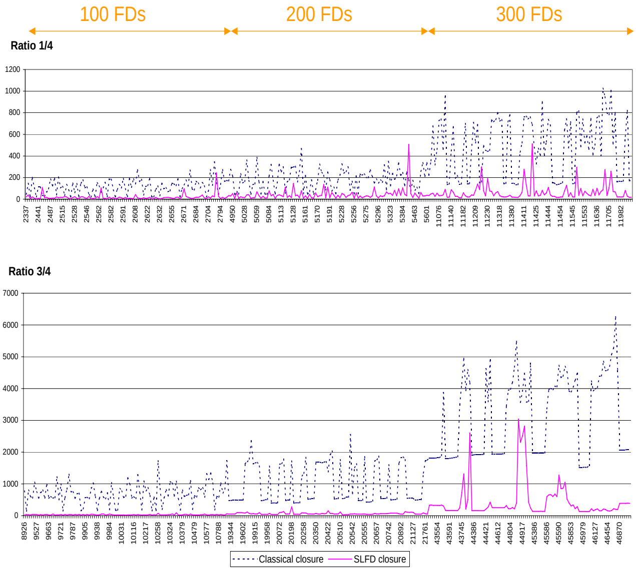
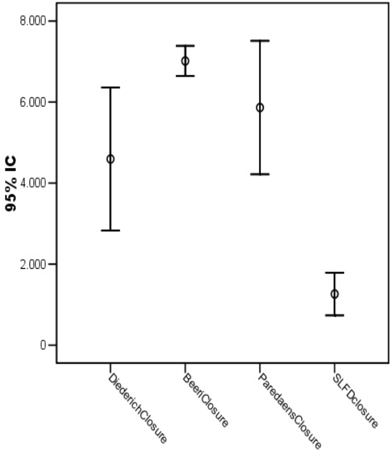
Automated reasoning





Fuzzy extension of the relational model

Fuzzyness on functional dependencies		Fuzzyness on data			Executable logic
		Classical data table	Table of fuzzy sets	Ranked data table	
	Functional dependency	[Codd, 1970] [Armstrong, 1974]			Simplification Logic [Mora et al, 2006]
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	Graded F.D. over dom. with similarities				
	Fuzzy functional dependency				

Fuzzy sets

- Fuzzy sets [Zadeh, 65]: $\mathcal{U} \to [0,1]$
- L-fuzzy sets [Goguen, 67]: $\mathcal{U} \to L$ where L is a complete lattice.
- Complete residuated lattices: $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1 \rangle$ with:
  - $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice.
  - $\langle L, \otimes, 1 \rangle$ is a commutative monoid.
  - $\otimes$ and $\to$ satisfy the adjointness property:

$$x \otimes y \leq z \text{ if and only if } x \leq y \to z$$

- A truth-stressing hedge * (shortly hedge): for all $x, y \in L$,

$$1^* = 1, \ x^* \leq x, \ (x \to y)^* \leq x^* \to y^*, \ \text{ and } \ x^{**} = x^*$$

Graded (fuzzy) sets

Having $\mathbf{L}$, we define the usual notions:

- An $\mathbf{L}$-set A in universe $\mathcal{U}$ is a mapping $A\colon \mathcal{U} \to L$ where

  $A(u)$ is "the degree in which u belongs to A".

- $\mathbf{L}^{\mathcal{U}}$ denotes the set of fuzzy sets in universe $\mathcal{U}$.
- Let $A, B \in \mathbf{L}^{\mathcal{U}}$ and $c \in L$.
  - The degree of inclusion of A in B is defined as:

    $$S(A,B) = \bigwedge_{u \in \mathcal{U}} (A(u) \to B(u))$$

    Note that $S(A,B) = 1$ iff $A(u) \leq B(u)$ for all $u \in \mathcal{U}$. In this case, we will write $A \subseteq B$.
  - $A \cup B$ is defined as $(A \cup B)(u) = A(u) \vee B(u)$ for all $u \in \mathcal{U}$.
  - $A \cap B$ is defined as $(A \cap B)(u) = A(u) \wedge B(u)$ for all $u \in \mathcal{U}$.
  - $c \otimes A$ is defined as $(c \otimes A)(u) = c \otimes A(u)$ for all $u \in \mathcal{U}$.

Fuzzy relations

A similarity relation in a non-empty set $\mathcal{U}$ is a mapping $\approx : \mathcal{U} \times \mathcal{U} \to L$ that satisfies:

- Reflexivity: $(a \approx a) = 1$ for all $a \in \mathcal{U}$.
- Symmetry: $(a \approx b) = (b \approx a)$ for all $a, b \in \mathcal{U}$.

A similarity relation is a fuzzy equivalence if it also satisfies:

- $\otimes$-transitivity: $(a \approx b) \otimes (b \approx c) \leq (a \approx c)$ for all $a, b, c \in \mathcal{U}$.

A fuzzy equality is a fuzzy equivalence in which $(a \approx b) = 1$ implies $a = b$.

Outline

First extension [Raju and Majumdar, 1988]

`job,experience` $\Rightarrow$ `salary` "Similar job and experience imply similar salary"

Let $\{(D_y, \approx_y) \mid y \in Y\}$ be a family of domains with similarity relations.

These relations can be extended to $D_A = \prod_{y \in A} D_y$, for all $A \subseteq Y$, as follows

$$(t_1 \approx_A t_2) = \bigwedge_{y \in A} (t_1[y] \approx_y t_2[y])$$

Definition

A data table $\mathcal{R}$ satisfies the functional dependency $A \Rightarrow B$ if, for all $t_1, t_2 \in \mathcal{R}$,

$$(t_1[A] \approx t_2[A]) \leq (t_1[B] \approx t_2[B])$$

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First extension [Raju and Majumdar, 1988]

- The functional dependency remains being crisp.
- Armstrong's axioms are sound and complete.
- Simplification logic and its automated deduction method can be used.

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Executable logic
Simplification Logic [Mora et al, 2006]
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Second approach

`job,experience` $\stackrel{c}{\Rightarrow}$ `salary` "Similar job and experience more or less imply similar salary"

Now, a functional dependency is a formula $A \Rightarrow B$ endowed with a grade of certainty $c \in L$.

A fuzzy theory is a fuzzy set in the language $\mathcal{L}$ (i.e. a map $T \in L^{\mathcal{L}}$) such that $T(A \Rightarrow B) = c \in L$.

We will denote it by $A \stackrel{c}{\Rightarrow} B$.

In a residuated lattice, $a \leq b$ iff $a \to b = 1$. This is the main idea to obtain a second approach:

Definition

A datatable $\mathcal{R}$ satisfies $A \stackrel{c}{\Rightarrow} B$ if, for all $t_1, t_2 \in \mathcal{R}$,

$$c \leq (t_1[A] \approx t_2[A]) \to (t_1[B] \approx t_2[B])$$

Or, equivalently, if

$$c \leq \bigwedge_{t_1, t_2 \in \mathcal{R}} (t_1[A] \approx t_2[A]) \to (t_1[B] \approx t_2[B])$$

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Or, equivalently, if

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Fuzzy logic

$\mathcal{R}$ satisfies $A \stackrel{c}{\Rightarrow} B$ if, for all two tuples $t_1, t_2 \in \mathcal{R}$,

$$c \leq \bigwedge_{t_1,t_2 \in \mathcal{R}} (t_1[A] \approx t_2[A]) \rightarrow (t_1[B] \approx t_2[B])$$

We can define the grade in which $\mathcal{R}$ satisfies $A \Rightarrow B$ as follows

$$||A \Rightarrow B||_{\mathcal{R}} = \bigwedge_{t_1,t_2 \in \mathcal{R}} (t_1[A] \approx t_2[A]) \rightarrow (t_1[B] \approx t_2[B])$$

and the set of models of a fuzzy theory $T \in L^{\mathcal{L}}$ as

$$Mod(T) = \{\mathcal{R} \mid T(A \Rightarrow B) \leq ||A \Rightarrow B||_{\mathcal{R}} \text{ for all } A, B \subseteq Y\}$$

Finally, $T \models A \stackrel{c}{\Rightarrow} B$ if $Mod(T) \subseteq Mod(A \stackrel{c}{\Rightarrow} B)$.

Fuzzy Simplification Logic [Cordero et al., 2010]

- **Axioms**: for all $B \subseteq A$, $\vdash A \stackrel{1}{\Rightarrow} B$.
- **Decomposition rule**: if $C \subseteq B$ and $c_2 \leq c_1$, $A \stackrel{c_1}{\Rightarrow} B \vdash A \stackrel{c_2}{\Rightarrow} C$.
- **Composition rule**: $A \stackrel{c_1}{\Rightarrow} B,\ C \stackrel{c_2}{\Rightarrow} D \ \vdash AC \stackrel{c_1 \wedge c_2}{\Rightarrow} BD$.
- **Simplification rule**: if $A \subseteq C$ and $A \cap B = \varnothing$ $A \stackrel{c_1}{\Rightarrow} B,\ C \stackrel{c_2}{\Rightarrow} D \ \vdash C \smallsetminus B \stackrel{c_1 \otimes c_2}{\Rightarrow} D \smallsetminus B$.

Soundness and completeness: For all fuzzy theory T and all graded formula $A \stackrel{c}{\Rightarrow} B$,

$$T \models A \stackrel{c}{\Rightarrow} B \text{ if and only if } T \vdash A \stackrel{c}{\Rightarrow} B$$

Moreover, the extension of the automated reasoning method has been provided in [Cordero et al., 2011].

Second approach

- [Yazici and Sozat, 1996] uses the Gödel product in $[0, 1]$.
- [Ben Yahia et al., 1999] uses the Łukasiewicz product in $[0, 1]$.

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	Fuzzy functional dependency				

Third approach [Belohlávek and Vychodil, 2006]

$$\{\texttt{job}, {}^{0.8}/\texttt{experience}\} \stackrel{0.9}{\Rightarrow} \{{}^{0.6}/\texttt{salary}\}$$

The following assertion is true to degree at least 0.9:

> *"Same job and experience similar to degree at least* 0.8 *imply similar salary to degree at least* 0.6*"*

In this case, a functional dependency is an expression $A \Rightarrow B$ where A and B are fuzzy sets

$$(t_1 \approx_A t_2) = \bigwedge_{y \in Y} A(y) \rightarrow (t_1[y] \approx_y t_2[y])$$

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The grade in which a data table $\mathcal{R}$ satisfies $A \Rightarrow B$ is

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Third approach [Belohlávek and Vychodil, 2006]

Lemma ([Belohlávek and Vychodil, 2005])

Let $A, B \in L^Y$, $c \in L$ *and* $\mathcal{R}$ *be a data table.*

$$c \leq ||A \Rightarrow B||_{\mathcal{D}} \text{ if and only if } ||A \Rightarrow c \otimes B||_{\mathcal{D}} = 1$$

and, therefore, any fuzzy theory T *is equivalent to the following crisp theory*

$$c(T) = \{A \Rightarrow T(A \Rightarrow B) \otimes B \mid A, B \in L^Y \text{ and } T(A \Rightarrow B) \otimes B \neq \varnothing\}$$

$\{\texttt{job}, {}^{0.8}/\texttt{experience}\} \overset{0.9}{\Rightarrow} \{{}^{0.6}/\texttt{salary}\}$ is equivalent to $\{\texttt{job}, {}^{0.8}/\texttt{experience}\} \Rightarrow \{{}^{0.9\otimes 0.6}/\texttt{salary}\}$

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Let $A, B, C, D \in L^Y$ and $c \in L$.

- Axioms: $\vdash AB \Rightarrow A$.
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Third approach

		Fuzzyness on data			
		Classical data table	Table of fuzzy sets	Ranked data table	Executable logic
Fuzzyness on functional dependencies	Functional dependency	[Codd, 1970] [Armstrong, 1974]			Simplification Logic [Mora et al, 2006]
	F.D. over domains with similarities	[Raju & Majumdar, 1988]			Simplification Logic [Mora et al, 2006]
	Graded F.D. over dom. with similarities	[Yazici & Sozat, 1996] [Ben Yahia et al, 1999]			Fuzzy Simplification Logic [Cordero et al, 2010]
	Fuzzy functional dependency	[Belohlávek & Vychodil, 2006]			

Outline

Tables of fuzzy sets

In [Buckles and Petri, 1982] and after in [Prade and Testemale, 1984], a fuzzy data table over a family of domains $\{D_y \mid y \in Y\}$ is defined as a subset

$$R \subseteq \prod_{y \in Y} L^{D_y}$$

The elements in each tuple are named "possibility distributions".

name	hair	skin	age	eyes	factor
John	black	dark	[30,40]	dark	10
Albert	clear	light	about-30	$\{^{1}/\text{blue}, ^{0.8}/\text{green}\}$	[40,50]
Mary	auburn	lightint	$\{26, ^{0.9}/27\}$	blue	50
Dave	quasi red	light	young	blue	about-50
Noa	White	dark	about-32	green	[25,35]

Tables of fuzzy sets vs (crisp) data tables

- Obviously, any (crisp) data table is a particular case of fuzzy data table.
- From the point of view of the theory of functional dependencies, fuzzy data tables can be considered particular cases of (crisp) data tables.

  If we provide a way to extend a similarity relation $\approx$ on a domain D to another similarity relation $\widehat{\approx}$ on L^D, then, by replacing the family of domains (with similarities)

$$\{(D_y, \approx_y) \mid y \in Y\} \quad \text{by} \quad \{(L^{D_y}, \widehat{\approx}_y) \mid y \in Y\}$$

  all the previous definitions of fuzzy functional dependencies can be extended.

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	Graded F.D. over dom. with similarities	[Yazici & Sozat, 1996] [Ben Yahia et al, 1999]	[Chen, 1991]		Fuzzy Simplification Logic [Cordero et al, 2010]
	Fuzzy functional dependency	[Belohlávek & Vychodil, 2006]	[Cubero & Vila, 1994]		

Ranked data table

[Baldwin, 1983] proposed an extension of the notion of datatable over $\{D_y \mid y \in Y\}$ as a fuzzy subset of the product

$$\mathcal{D}\colon \prod_{y\in Y} D_y \to L$$

This notion was used also in [Raju and Majumdar, 1988] and [Tyagi et al, 2005].
Recently, [Belohlávek and Vychodil, 2006] have given a reasonable semantic for this kind of data tables.

$\mathcal{D}$(t)	name	hair	skin	age	eyes	factor
1.0	John	Black	dark	34	Brown	10
0.8	Albert	Brown	light	32	Blue	50
0.6	Mary	Auburn	lig-int	29	Blue	50
0.4	Dave	Red	light	26	Blue	50
0.1	Noa	White	dark	44	Green	30

It may be seen as an answer to a similarity query "show all persons with age approximately 34".

Fuzzy functional dependencies on ranked data tables

The most general definition of fuzzy functional dependency has been introduced by [Belohlávek and Vychodil, 2006]
Given a family of domains with similarities $\{(D_y, \approx_y) \mid y \in Y\}$ and a ranked data table

$$\mathcal{D}\colon \prod_{y \in Y} D_y \to L$$

the relative similarity relation is defined as follows

$$(t_1 \approx_{\mathcal{D}} t_2) = \big(\mathcal{D}(t_1) \otimes \mathcal{D}(t_1)\big) \to \bigwedge_{y \in Y} \big(A(y) \to (t_1[y] \approx_y t_2[y])\big)$$

Definition

The grade in which $\mathcal{D}$ satisfies $A \Rightarrow B$ is

$$||A \Rightarrow B||_{\mathcal{D}} = \bigwedge_{t_1, t_2} (t_1[A] \approx_{\mathcal{D}} t_2[A])^* \to (t_1[B] \approx_{\mathcal{D}} t_2[B])$$

Ranked data tables

		Fuzzyness on data			Executable logic
		Classical data table	Table of fuzzy sets	Ranked data table	
Fuzzyness on functional dependencies	Functional dependency	[Codd, 1970] [Armstrong, 1974]	[Buckles & Petri, 1982] [Prade & Testemale, 1984]	[Baldwin & Zhou, 1983] [Raju & Majumdar, 1988]	Simplification Logic [Mora et al, 2006]
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Outline

New axiomatic system

Our starting point is the axiomatic system proposed by R. Belohlavek and V. Vychodil.

Definition

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Difference: A new operation over fuzzy sets

- The paradigm of the simplification logics is to infer implicit information via redundancy removing.
- When we work with dependencies $A \Rightarrow B$ in which A and B are (crisp) sets we use the difference of sets $A \smallsetminus B$.
- So, we need to extend this difference to fuzzy sets.
- Our aim is to define a sound and complete axiomatic system based on simplification.
- There exist different ways to extend it. What is appropriate to reach our objective?
- It is necessary that the following equalities hold:

$$A \smallsetminus B \subseteq A \qquad \text{and} \qquad (A \smallsetminus B) \cup B = A \cup B \qquad \text{for all } A, B \in \mathbf{L}^{\mathcal{U}}.$$

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Structures of degrees

- We consider an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, \smallsetminus, ^*, 0, 1\rangle$ such that:
  - $\langle L, \wedge, \vee, \otimes, \rightarrow, ^*, 0, 1\rangle$ is a complete residuated lattice with hedge.
  - For all $x, y, z \in L$, $x \smallsetminus y \leq z$ if and only if $x \leq y \vee z$.
- Consequently, $\langle L, \wedge, \vee, \smallsetminus, 0, 1\rangle$ is a Browerian algebra (dual to a Heyting algebra) and,
- so, $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice.

Example

Let us consider the subset of the unit interval $\{0, 0.1, 0.2, \ldots, 0.9, 1\}$ with the natural ordering, the Łukasiewiz adjoint par, the difference and the hedge given by

$$x \otimes y = \max\{x + y - 1, 0\} \qquad x \smallsetminus y = \begin{cases} x & \text{if } x > y, \\ 0 & \text{otherwise.} \end{cases} \qquad x^* = \begin{cases} 1 & \text{if } x = 1, \\ 0.5 & \text{if } 0.5 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$x \rightarrow y = \min\{1 - x + y, 1\}$$

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New axiomatic system

In the new syntactico-semantically complete axiomatic system, rule Cut is replace by a new rule named *rule of simplification*.

Definition

Let $A, B, C, D \in \mathbf{L}^Y$ and $c \in L$.

- Axioms: $\vdash AB \Rightarrow A$.
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The new system is called FALS (Fuzzy Attribute Logic with the rule of Simplification)

Lemma

The following inference rules are derived: Let $A, B, C, D \in \mathbf{L}^Y$.

- *Decomposition rule:* $\{A \Rightarrow BC\} \vdash A \Rightarrow B$.
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Some derived equivalences

The importance of these inference rules is that they can be easily extended to obtain a set of equivalencies that are focussed on removing redundant information in the theories.

Theorem

Let $A, B, C, D \in \mathbf{L}^Y$.

1. *Decomposition Eq.:* $\{A \Rightarrow B\} \equiv \{A \Rightarrow B \smallsetminus A\}$
2. *Union Eq.:* $\{A \Rightarrow B, A \Rightarrow C\} \equiv \{A \Rightarrow BC\}$
3. *Simplification Eq.: If* $A \subseteq C$ *then* $\{A \Rightarrow B, C \Rightarrow D\} \equiv \{A \Rightarrow B, A(C \smallsetminus B) \Rightarrow D \smallsetminus B\}$

The automated reasoning method

Theorem

Let $A, B \in \mathbf{L}^Y$, $c \in L$ *and* $T \in \mathbf{L}^{\mathcal{L}}$.

$$T \vdash A \overset{c}{\Rightarrow} B \text{ if and only if } \{\varnothing \Rightarrow A\} \cup c(T) \vdash \varnothing \Rightarrow c \otimes B$$

Theorem

For all $A, B, C \in \mathbf{L}^Y$, *if* $A' = A(S(B, A)^* \otimes C)$ *then*

$$\{\varnothing \Rightarrow A, B \Rightarrow C\} \equiv \{\varnothing \Rightarrow A', B - A' \Rightarrow C - A'\}$$

Particularly,

1. *if* $B \smallsetminus A' = \varnothing$ *then* $\{\varnothing \Rightarrow A, B \Rightarrow C\} \equiv \{\varnothing \Rightarrow A'C\}$.
2. *if* $C \smallsetminus A' = \varnothing$ *then* $\{\varnothing \Rightarrow A, B \Rightarrow C\} \equiv \{\varnothing \Rightarrow A'\}$.

This equivalency will be named *Generalized Simplification Equivalence* and denoted **(gSiEq)**. The first particular case, in which we also apply the union equivalence, will be denoted by **(gSiUnEq)** and the second one will be denoted **(gSiAxEq)** because an axiom has been removed.

Example

We consider the truthfulness structure described in Example 8. Let T be the following fuzzy theory.

$$
\begin{array}{rlll}
T = \{ & \{0.4/a, 0.6/c\} & \stackrel{0.6}{\Rightarrow} \{0.8/c, 0.5/d, 0.6/e, 0.7/f\}, & \boxed{1} \\
& \{0.2/d, 0.3/f\} & \stackrel{0.9}{\Rightarrow} \{1/d,\ 0.6/e,\ 0.9/g\}, & \boxed{2} \\
& \{0.4/d, 0.5/e\} & \stackrel{0.8}{\Rightarrow} \{0.6/h, 0.2/d\}, & \boxed{3} \\
& \{0.6/d, 0.4/i\} & \stackrel{1}{\Rightarrow} \{0.7/a, 0.7/d\}, & \boxed{4} \\
& \{0.3/c, 0.4/e\} & \stackrel{1}{\Rightarrow} \{0.2/h\}, & \boxed{5} \\
& \{0.4/c, 0.6/h\} & \stackrel{0.6}{\Rightarrow} \{0.3/b, 0.7/e, 0.8/i\}, & \boxed{6} \\
& \{0.2/g\} & \stackrel{0.6}{\Rightarrow} \{0.7/a, 0.4/d\}, & \boxed{7} \\
& \{0.6/c, 0.5/d\} & \stackrel{0.8}{\Rightarrow} \{0.4/e\}\} & \boxed{8}
\end{array}
$$

and we want to check if

$$T \vdash \{0.2/c, 0.6/f\} \stackrel{0.8}{\Rightarrow} \{0.5/a, 0.5/d, 0.6/g, 0.6/h\}$$

Example

By Theorem 12, this problem is equivalent to the following:

$$\{\varnothing \Rightarrow A\} \cup c(T) \vdash \{\varnothing \Rightarrow \{0.3/a, 0.3/d, 0.4/g, 0.4/h\}\}$$

being $\{\varnothing \Rightarrow A\}$ and $c(T)$ the following formula and (crisp) theory

$$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \Rightarrow \{0.2/c, 0.6/f\}\} \qquad \boxed{\text{Guide}}$$

$$\begin{aligned} c(T) = \{ \quad & \{0.4/a, 0.6/c\} \Rightarrow \{0.4/c, 0.1/d, 0.2/e, 0.3/f\} & \boxed{1} \\ & \{0.2/d, 0.3/f\} \Rightarrow \{0.9/d, 0.5/e, 0.8/g\} & \boxed{2} \\ & \{0.4/d, 0.5/e\} \Rightarrow \{0.4/h\} & \boxed{3} \\ & \{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\} & \boxed{4} \\ & \{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\} & \boxed{5} \\ & \{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\} & \boxed{6} \\ & \{0.2/g\} \Rightarrow \{0.3/a\} & \boxed{7} \\ & \{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\} \ \} & \boxed{8} \end{aligned}$$

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$\{\varnothing \Rightarrow A\} = \{$	$\varnothing$	$\Rightarrow \{0.2/c, 0.6/f\}\}$	Guide
$c(T) = \{$	$\{0.4/a, 0.6/c\}$	$\Rightarrow \{0.4/c, 0.1/d, 0.2/e, 0.3/f\}$	1
	$\{0.2/d, 0.3/f\}$	$\Rightarrow \{0.9/d, 0.5/e, 0.8/g\}$	2
	$\{0.4/d, 0.5/e\}$	$\Rightarrow \{0.4/h\}$	3
	$\{0.6/d, 0.4/i\}$	$\Rightarrow \{0.7/a, 0.7/d\}$	4
	$\{0.3/c, 0.4/e\}$	$\Rightarrow \{0.2/h\}$	5
	$\{0.4/c, 0.6/h\}$	$\Rightarrow \{0.3/e, 0.4/i\}$	6
	$\{0.2/g\}$	$\Rightarrow \{0.3/a\}$	7
	$\{0.6/c, 0.5/d\}$	$\Rightarrow \{0.2/e\}\ \}$	8

Applying to every formula: $(\mathbf{DeEq})$: $\{A \Rightarrow B\} \equiv \{A \Rightarrow B \smallsetminus A\}$

Example

By Theorem 12, this problem is equivalent to the following:

$$\{\varnothing \Rightarrow A\} \cup c(T) \;\vdash\; \{\varnothing \Rightarrow \{0.3/a, 0.3/d, 0.4/g, 0.4/h\}\}$$

being $\{\varnothing \Rightarrow A\}$ and $c(T)$ the following formula and (crisp) theory

$$\{\varnothing \Rightarrow A\} = \{\ \varnothing \Rightarrow \{0.2/c, 0.6/f\}\} \qquad \boxed{\text{Guide}}$$

$$\begin{aligned} c(T) = \{\ & \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e, 0.3/f\} && \boxed{1} \\ & \{0.2/d, 0.3/f\} \Rightarrow \{0.9/d, 0.5/e, 0.8/g\} && \boxed{2} \\ & \{0.4/d, 0.5/e\} \Rightarrow \{0.4/h\} && \boxed{3} \\ & \{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\} && \boxed{4} \\ & \{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\} && \boxed{5} \\ & \{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\} && \boxed{6} \\ & \{0.2/g\} \Rightarrow \{0.3/a\} && \boxed{7} \\ & \{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\}\ \} && \boxed{8} \end{aligned}$$

Applying to every formula: $(\mathbf{DeEq})$: $\{A \Rightarrow B\} \equiv \{A \Rightarrow B \smallsetminus A\}$

Example

$$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.6/f\}\} \qquad \boxed{\text{Guide}}$$

$$\begin{aligned} T = \{ \quad & \{0.4/a, 0.6/c\} && \Rightarrow \{0.1/d, 0.2/e, 0.3/f\} && \boxed{1} \\ & \{0.2/d, 0.3/f\} && \Rightarrow \{0.9/d, 0.5/e, 0.8/g\} && \boxed{2} \\ & \{0.4/d, 0.5/e\} && \Rightarrow \{0.4/h\} && \boxed{3} \\ & \{0.6/d, 0.4/i\} && \Rightarrow \{0.7/a, 0.7/d\} && \boxed{4} \\ & \{0.3/c, 0.4/e\} && \Rightarrow \{0.2/h\} && \boxed{5} \\ & \{0.4/c, 0.6/h\} && \Rightarrow \{0.3/e, 0.4/i\} && \boxed{6} \\ & \{0.2/g\} && \Rightarrow \{0.3/a\} && \boxed{7} \\ & \{0.6/c, 0.5/d\} && \Rightarrow \{0.2/e\} \ \} && \boxed{8} \end{aligned}$$

(gSiEq):

$$\{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.6/f\}, \quad \{0.4/a, 0.6/c\} \quad \Rightarrow \{0.1/d, 0.2/e, 0.3/f\} \quad \} \equiv$$

$$\equiv \{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.6/f\}, \quad \{0.4/a, 0.6/c\} \quad \Rightarrow \{0.1/d, 0.2/e\} \quad \}$$

Example

$$\{\varnothing \Rightarrow A\} = \{\ \varnothing \Rightarrow \{0.2/c, 0.6/f\}\}$$ Guide

$$
\begin{aligned}
T = \{\ & \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e, 0.3/f\} & \boxed{1}\\
& \{0.2/d, 0.3/f\} \Rightarrow \{0.9/d, 0.5/e, 0.8/g\} & \boxed{2}\\
& \{0.4/d, 0.5/e\} \Rightarrow \{0.4/h\} & \boxed{3}\\
& \{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\} & \boxed{4}\\
& \{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\} & \boxed{5}\\
& \{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\} & \boxed{6}\\
& \{0.2/g\} \Rightarrow \{0.3/a\} & \boxed{7}\\
& \{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\}\ \} & \boxed{8}
\end{aligned}
$$

(gSiEq):

$$\{\ \varnothing \Rightarrow \{0.2/c, 0.6/f\},\quad \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e, 0.3/f\}\ \} \equiv$$

$$\equiv \{\ \varnothing \Rightarrow \{0.2/c, 0.6/f\},\quad \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\}\ \}$$

Example

$$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \Rightarrow \{0.2/c, 0.6/f\}\}$$ Guide

$$
\begin{array}{lll}
T = \{ & \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\} & 1 \\
& \{0.2/d, 0.3/f\} \Rightarrow \{0.9/d, 0.5/e, 0.8/g\} & 2 \\
& \{0.4/d, 0.5/e\} \Rightarrow \{0.4/h\} & 3 \\
& \{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\} & 4 \\
& \{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\} & 5 \\
& \{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\} & 6 \\
& \{0.2/g\} \Rightarrow \{0.3/a\} & 7 \\
& \{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\} \} & 8
\end{array}
$$

(gSiEq):

$$\{ \quad \varnothing \Rightarrow \{0.2/c, 0.6/f\}, \quad \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e, 0.3/f\} \quad \} \equiv$$

$$\equiv \{ \quad \varnothing \Rightarrow \{0.2/c, 0.6/f\}, \quad \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\} \quad \}$$

Example

$$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.6/f\}\}$$ Guide

$$
\begin{aligned}
T = \{ \quad & \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\} & 1 \\
& \{0.2/d, 0.3/f\} \Rightarrow \{0.9/d, 0.5/e, 0.8/g\} & 2 \\
& \{0.4/d, 0.5/e\} \Rightarrow \{0.4/h\} & 3 \\
& \{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\} & 4 \\
& \{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\} & 5 \\
& \{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\} & 6 \\
& \{0.2/g\} \Rightarrow \{0.3/a\} & 7 \\
& \{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\} \} & 8
\end{aligned}
$$

(gSiUnEq):

$$
\begin{aligned}
& \{ \ \varnothing \Rightarrow \{0.2/c, 0.6/f\}, \quad \{0.2/d, 0.3/f\} \Rightarrow \{0.9/d, 0.5/e, 0.8/g\} \ \} \equiv \\
\equiv & \{ \ \varnothing \Rightarrow \{0.2/c, 0.4/d, 0.6/f, 0.3/g\}, \quad \varnothing \Rightarrow \{0.9/d, 0.5/e, 0.8/g\} \ \} \equiv \\
\equiv & \{ \ \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g\} \ \}
\end{aligned}
$$

Example

$$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.6/f\}\}$$

Guide

$$
\begin{aligned}
T = \{ \quad & \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\} & 1\\
& \{0.2/d, 0.3/f\} \Rightarrow \{0.9/d, 0.5/e, 0.8/g\} & 2\\
& \{0.4/d, 0.5/e\} \Rightarrow \{0.4/h\} & 3\\
& \{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\} & 4\\
& \{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\} & 5\\
& \{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\} & 6\\
& \{0.2/g\} \Rightarrow \{0.3/a\} & 7\\
& \{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\} \} & 8
\end{aligned}
$$

(gSiUnEq):

$$\{ \quad \varnothing \Rightarrow \{0.2/c, 0.6/f\}, \quad \{0.2/d, 0.3/f\} \Rightarrow \{0.9/d, 0.5/e, 0.8/g\} \quad \} \equiv$$

$$\equiv \{ \quad \varnothing \Rightarrow \{0.2/c, 0.4/d, 0.6/f, 0.3/g\}, \quad \varnothing \Rightarrow \{0.9/d, 0.5/e, 0.8/g\} \quad \} \equiv$$

$$\equiv \{ \quad \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g\} \quad \}$$

Example

$$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g\}\}$$ Guide

$$T = \{ \quad \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e \quad \}$$ 1

2

$$\{0.4/d, 0.5/e\} \Rightarrow \{0.4/h\}$$ 3

$$\{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\}$$ 4

$$\{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\}$$ 5

$$\{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\}$$ 6

$$\{0.2/g\} \Rightarrow \{0.3/a\}$$ 7

$$\{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\} \}$$ 8

(gSiUnEq):

$$\{ \quad \varnothing \Rightarrow \{0.2/c, 0.6/f\}, \qquad \{0.2/d, 0.3/f\} \Rightarrow \{0.9/d, 0.5/e, 0.8/g\} \quad \} \equiv$$

$$\equiv \{ \quad \varnothing \Rightarrow \{0.2/c, 0.4/d, 0.6/f, 0.3/g\}, \qquad \varnothing \Rightarrow \{0.9/d, 0.5/e, 0.8/g\} \quad \} \equiv$$

$$\equiv \{ \quad \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g\} \quad \}$$

Example

$$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g\}\}$$

Guide

$$T = \{ \quad \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e \quad \}$$

$$\{0.4/d, 0.5/e\} \Rightarrow \{0.4/h\}$$

$$\{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\}$$

$$\{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\}$$

$$\{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\}$$

$$\{0.2/g\} \Rightarrow \{0.3/a\}$$

$$\{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\} \}$$

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(gSiAxEq):

$$\{ \quad \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g\}, \quad \{0.4/d, 0.5/e\} \Rightarrow \{0.4/h\} \quad \} \equiv$$

$$\equiv \{ \quad \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \quad \varnothing \Rightarrow \varnothing \quad \} \equiv$$

$$\equiv \{ \quad \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\} \quad \}$$

Example

$$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g\}\}$$

Guide

$$
\begin{aligned}
T = \{ \quad & \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\ \} & 1\\
& & 2\\
& \{0.4/d, 0.5/e\} \Rightarrow \{0.4/h\} & 3\\
& \{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\} & 4\\
& \{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\} & 5\\
& \{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\} & 6\\
& \{0.2/g\} \Rightarrow \{0.3/a\} & 7\\
& \{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\} \ \} & 8
\end{aligned}
$$

(gSiAxEq):

$$
\begin{aligned}
& \{ \quad \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g\}, \quad \{0.4/d, 0.5/e\} \Rightarrow \{0.4/h\} \quad \} \equiv \\
\equiv & \{ \quad \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \quad \varnothing \Rightarrow \varnothing \quad \} \equiv \\
\equiv & \{ \quad \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\} \quad \}
\end{aligned}
$$

Example

$$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$ Guide

$$T = \{ \quad \{0.4/a, 0.6/c\} \quad \Rightarrow \{0.1/d, 0.2/e \quad \} $$ 1

2

3

$$\{0.6/d, 0.4/i\} \quad \Rightarrow \{0.7/a, 0.7/d\}$$ 4

$$\{0.3/c, 0.4/e\} \quad \Rightarrow \{0.2/h\}$$ 5

$$\{0.4/c, 0.6/h\} \quad \Rightarrow \{0.3/e, 0.4/i\}$$ 6

$$\{0.2/g\} \quad \Rightarrow \{0.3/a\}$$ 7

$$\{0.6/c, 0.5/d\} \quad \Rightarrow \{0.2/e\} \}$$ 8

(gSiAxEq):

$$\{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g\}, \qquad \{0.4/d, 0.5/e\} \quad \Rightarrow \{0.4/h\} \quad \} \equiv$$

$$\equiv \{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \qquad \varnothing \quad \Rightarrow \varnothing \quad \} \equiv$$

$$\equiv \{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\} \quad \}$$

Example

$$\{\varnothing \Rightarrow A\} = \{\quad \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$ Guide

$$T = \{\quad \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\}$$ 1

2

3

$$\{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\}$$ 4

$$\{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\}$$ 5

$$\{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\}$$ 6

$$\{0.2/g\} \Rightarrow \{0.3/a\}$$ 7

$$\{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\}\ \}$$ 8

(gSiEq):

$$\{\quad \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \quad \{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\}\quad \} \equiv$$

$$\equiv \{\quad \varnothing \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \quad \{0.4/i\} \Rightarrow \{0.7/a\}\quad \}$$

Example

$$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Guide

$$
\begin{aligned}
T = \{ \quad & \{0.4/a, 0.6/c\} && \Rightarrow \{0.1/d, 0.2/e \quad\} && 1 \\
& && && 2 \\
& && && 3 \\
& \{0.6/d, 0.4/i\} && \Rightarrow \{0.7/a, 0.7/d\} && 4 \\
& \{0.3/c, 0.4/e\} && \Rightarrow \{0.2/h\} && 5 \\
& \{0.4/c, 0.6/h\} && \Rightarrow \{0.3/e, 0.4/i\} && 6 \\
& \{0.2/g\} && \Rightarrow \{0.3/a\} && 7 \\
& \{0.6/c, 0.5/d\} && \Rightarrow \{0.2/e\} \ \} && 8
\end{aligned}
$$

(gSiEq):

$$\{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \qquad \{0.6/d, 0.4/i\} \quad \Rightarrow \{0.7/a, 0.7/d\} \quad \} \equiv$$

$$\equiv \{ \quad \varnothing \quad \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \qquad \{0.4/i\} \quad \Rightarrow \{0.7/a\} \qquad \}$$

Example

$$\{\varnothing \Rightarrow A\} = \{\quad \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Guide

$$
\begin{aligned}
T = \{\quad & \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\} && 1\\
& && 2\\
& && 3\\
& \{0.4/i\} \Rightarrow \{0.7/a\} && 4\\
& \{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\} && 5\\
& \{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\} && 6\\
& \{0.2/g\} \Rightarrow \{0.3/a\} && 7\\
& \{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\}\ \} && 8
\end{aligned}
$$

(gSiEq):

$$\{\quad \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \qquad \{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\}\quad \} \equiv$$

$$\equiv \{\quad \varnothing \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \qquad \{0.4/i\} \Rightarrow \{0.7/a\}\quad \}$$

Example

$$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Guide

$$T = \{ \quad \{0.4/a, 0.6/c\} \quad \Rightarrow \{0.1/d, 0.2/e \quad \}$$

$$\{ \quad 0.4/i\} \quad \Rightarrow \{0.7/a \quad \}$$

$$\{0.3/c, 0.4/e\} \quad \Rightarrow \{0.2/h\}$$

$$\{0.4/c, 0.6/h\} \quad \Rightarrow \{0.3/e, 0.4/i\}$$

$$\{0.2/g\} \quad \Rightarrow \{0.3/a\}$$

$$\{0.6/c, 0.5/d\} \quad \Rightarrow \{0.2/e\} \}$$

1 2 3 4 5 6 7 8

(gSiAxEq):

$$\{ \quad \varnothing \quad \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \quad \{0.3/c, 0.4/e\} \quad \Rightarrow \{0.2/h\} \quad \} \equiv$$

$$\equiv \{ \quad \varnothing \quad \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\} \quad \}$$

Example

$$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Guide

$$T = \{ \quad \{0.4/a, 0.6/c\} \quad \Rightarrow \{0.1/d, 0.2/e \quad\quad \}$$

$$\{ \quad\quad 0.4/i\} \quad \Rightarrow \{0.7/a \quad\quad \}$$

$$\{0.3/c, 0.4/e\} \quad \Rightarrow \{0.2/h\}$$

$$\{0.4/c, 0.6/h\} \quad \Rightarrow \{0.3/e, 0.4/i\}$$

$$\{0.2/g\} \quad \Rightarrow \{0.3/a\}$$

$$\{0.6/c, 0.5/d\} \quad \Rightarrow \{0.2/e\} \; \}$$

1 2 3 4 5 6 7 8

(gSiAxEq):

$$\{ \quad \varnothing \quad \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \quad \{0.3/c, 0.4/e\} \quad \Rightarrow \{0.2/h\} \quad \} \equiv$$

$$\equiv \{ \quad \varnothing \quad \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\} \quad \}$$

Example

$$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Guide

$$T = \{ \quad \{0.4/a, 0.6/c\} \quad \Rightarrow \{0.1/d, 0.2/e \quad \}$$

$$\{ \quad 0.4/i\} \quad \Rightarrow \{0.7/a \quad \}$$

$$\{0.4/c, 0.6/h\} \quad \Rightarrow \{0.3/e, 0.4/i\}$$

$$\{0.2/g\} \quad \Rightarrow \{0.3/a\}$$

$$\{0.6/c, 0.5/d\} \quad \Rightarrow \{0.2/e\} \}$$

1 2 3 4 5 6 7 8

(gSiAxEq):

$$\{ \quad \varnothing \quad \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \quad \{0.3/c, 0.4/e\} \quad \Rightarrow \{0.2/h\} \quad \} \equiv$$

$$\equiv \{ \quad \varnothing \quad \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\} \quad \}$$

Example

$$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Guide

$$T = \{ \quad \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e \quad \}$$

$$\{ \quad 0.4/i\} \Rightarrow \{0.7/a \quad \}$$

$$\{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\}$$

$$\{0.2/g\} \Rightarrow \{0.3/a\}$$

$$\{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\} \}$$

1 2 3 4 5 6 7 8

(gSiEq):

$$\{ \quad \varnothing \quad \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \quad \{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\} \quad \} \equiv$$

$$\equiv \{ \quad \varnothing \quad \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \quad \{0.4/c, 0.6/h\} \Rightarrow \{0.4/i\} \quad \}$$

Example

$$\{\varnothing \Rightarrow A\} = \{\ \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Guide

$$T = \{\ \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\ \}$$

$$\{\ 0.4/i\} \Rightarrow \{0.7/a\ \}$$

$$\{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\}$$

$$\{0.2/g\} \Rightarrow \{0.3/a\}$$

$$\{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\}\ \}$$

1
2
3
4
5
6
7
8

(gSiEq):

$$\{\ \varnothing \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\},\ \{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\}\ \} \equiv$$

$$\equiv \{\ \varnothing \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\},\ \{0.4/c, 0.6/h\} \Rightarrow \{0.4/i\}\ \}$$

Example

$$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Guide

$$T = \{ \quad \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e \}$$

$$\{ \quad 0.4/i\} \Rightarrow \{0.7/a \}$$

$$\{0.4/c, 0.6/h\} \Rightarrow \{ \quad 0.4/i\}$$

$$\{0.2/g\} \Rightarrow \{0.3/a\}$$

$$\{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\} \}$$

1 2 3 4 5 6 7 8

(gSiEq):

$$\{ \quad \varnothing \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \quad \{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\} \quad \} \equiv$$

$$\equiv \{ \quad \varnothing \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \quad \{0.4/c, 0.6/h\} \Rightarrow \{0.4/i\} \quad \}$$

Example

$$\{\varnothing \Rightarrow A\} = \{\quad \varnothing \quad \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Guide

$$T = \{\quad \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e \quad\}$$ 1

2

3

$$\{\quad 0.4/i\} \Rightarrow \{0.7/a \quad\}$$ 4

5

$$\{0.4/c, 0.6/h\} \Rightarrow \{\quad 0.4/i\}$$ 6

$$\{0.2/g\} \Rightarrow \{0.3/a\}$$ 7

$$\{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\}\ \}$$ 8

(gSiAxEq):

$$\{\quad \varnothing \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \quad \{0.2/g\} \Rightarrow \{0.3/a\} \quad\} \equiv$$

$$\equiv \{\quad \varnothing \Rightarrow \{0.3/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\} \quad\}$$

Example

$$\{\varnothing \Rightarrow A\} = \{\quad \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

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1 2 3 4 5 6 7 8

(gSiAxEq):

$$\{\quad \varnothing \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \quad \{0.2/g\} \Rightarrow \{0.3/a\} \quad \} \equiv$$

$$\equiv \{\quad \varnothing \Rightarrow \{0.3/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\} \quad \}$$

Example

$$\{\varnothing \Rightarrow A\} = \{\quad \varnothing \quad \Rightarrow \{0.3/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

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1 2 3 4 5 6 7 8

(gSiAxEq):

$$\{\quad \varnothing \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \quad \{0.2/g\} \Rightarrow \{0.3/a\}\quad\} \equiv$$

$$\equiv \{\quad \varnothing \Rightarrow \{0.3/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\quad\}$$

Example

$$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \quad \Rightarrow \{0.3/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Guide

$$T = \{ \quad \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\}$$

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$$\{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\} \}$$

1 2 3 4 5 6 7 8

Conclusion:

$$T \vdash \{0.2/a, 0.3/f\} \stackrel{0.8}{\Rightarrow} \{0.5/a, 0.5/d, 0.6/g, 0.6/h\}$$

because

$$0.8 \otimes \{0.5/a, 0.5/d, 0.6/g, 0.6/h\} = \{0.3/a, 0.3/d, 0.4/g, 0.4/h\} \subseteq \{0.3/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}$$

Conclusion

		Fuzzyness on data			Executable logic
		Classical data table	Table of fuzzy sets	Ranked data table	
Fuzzyness on functional dependencies	Functional dependency	[Codd, 1970] [Armstrong, 1974]	[Buckles & Petri, 1982] [Prade & Testemale, 1984]	[Baldwin & Zhou, 1983] [Raju & Majumdar, 1988]	Simplification Logic [Mora et al, 2006]
	F.D. over domains with similarities	[Raju & Majumdar, 1988]	[Liu, 1994] [Saxena & Tyagi, 1995]	[Raju & Majumdar, 1988] [Tyagi et al, 2005]	Simplification Logic [Mora et al, 2006]
	Graded F.D. over dom. with similarities	[Yazici & Sozat, 1996] [Ben Yahia et al, 1999]	[Chen, 1991]	[Cordero et al, 2011]	Fuzzy Simplification Logic [Cordero et al, 2010]
	Fuzzy functional dependency	[Belohlávek & Vychodil, 2006]	[Cubero & Vila, 1994]	[Belohlávek & Vychodil, 2006]	Fuzzy Attribute Logic based on Simplification

Relational model of data over domains with similarities

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Abiteboul, S. (2005).
The lowell database research self-assessment comm.
ACM, 48(5):111–118.

Armstrong, W. W. (1974).
Dependency structures of data base relationships.
In *IFIP Congress*, pages 580–583.

Baldwin, J. (1983).
Knowledge engineering using a fuzzy relational inference language.
In *Proc. IFAC Conf. on Fuzzy Information, Knowledge Representation, and Decision Processes*, pages 15–23, Marseille, France.

Belohlávek, R. and Vychodil, V. (2005).
Axiomatizations of fuzzy attribute logic.
In *IICAI'05*, pages 2178–2193.

Belohlávek, R. and Vychodil, V. (2006).
Data tables with similarity relations: Functional dependencies, complete rules and non-redundant bases.
In Lee, M.-L., Tan, K.-L., and Wuwongse, V., editors, *DASFAA*, volume 3882 of *Lecture Notes in Computer Science*, pages 644–658. Springer.

Ben Yahia, S., Ounalli, H., and Jaoua, A. (1999).
Extension of classical functional dependency: Dynamic fuzzy functional dependency.