Eduard Bartl: Minimal Solutions of Several Types of Fuzzy Relational Equations

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Preliminaries

We need the following concepts introduced in [2].

Aggregation Structure

Definition 1. A sup-preserving aggregation structure (aggregation structure, for short) $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box \rangle$, where $\mathbf{L}_i = \langle L_i, \leq_i \rangle$, $i = 1, 2, 3$, are complete lattices and $\Box : L_1 \times L_2 \to L_3$ is a function which commutes with suprema in both arguments. Define operations $\circ_{\Box} : L_1 \times L_3 \to L_2$, ${}_{\Box}\!\circ : L_3 \times L_2 \to L_1$ by

$$a_1 \circ_{\Box} a_3 = \bigvee\nolimits_2 \{a_2 \,|\, a_1 \Box a_2 \leq_3 a_3\},$$
$$a_3 \,{}_{\Box}\!\circ\, a_2 = \bigvee\nolimits_1 \{a_1 \,|\, a_1 \Box a_2 \leq_3 a_3\}.$$

Moreover, define operation $\overset{\mathrm{op}}{\Box}\!\circ : L_3 \times L_2 \to L_1$ (needed only for scalar-by-scalar equations) by

$$a_3 \overset{\mathrm{op}}{\Box}\!\circ a_2 = \begin{cases} \bigwedge_1 \{a_1 \,|\, a_1 \Box a_2 \geq_3 a_3\}, & \text{if exists } a_1 \in L_1 : a_1 \Box a_2 \geq_3 a_3, \\ 0_2, & \text{otherwise.} \end{cases}$$

Consider two important examples of aggreg. struct. In both cases, $\langle L, \wedge, \vee, \otimes, \to, 0, 1 \rangle$ is a complete residuated lattice. $L_i = L$ and $\leq_i$ is either $\leq$ or the dual of $\leq$.

Example 1. $\mathbf{L}_1 = \langle L, \leq \rangle$, $\mathbf{L}_2 = \langle L, \leq \rangle$, $\mathbf{L}_3 = \langle L, \leq \rangle$, $\Box = \otimes$:

$$a_1 \circ_{\Box} a_3 = \bigvee \{a_2 \,|\, a_1 \otimes a_2 \leq a_3\} = a_1 \to a_3,$$
$$a_3 \,{}_{\Box}\!\circ\, a_2 = \bigvee \{a_1 \,|\, a_1 \otimes a_2 \leq a_3\} = a_3 \leftarrow a_2.$$

Example 2. $\mathbf{L}_1 = \langle L, \leq \rangle$, $\mathbf{L}_2 = \langle L, \leq^{-1} \rangle$, $\mathbf{L}_3 = \langle L, \leq^{-1} \rangle$, $\Box = \to$:

$$a_1 \circ_{\Box} a_3 = \bigwedge \{a_2 \,|\, a_1 \to a_2 \geq a_3\} = a_1 \otimes a_3,$$
$$a_3 \,{}_{\Box}\!\circ\, a_2 = \bigvee \{a_1 \,|\, a_1 \to a_2 \geq a_3\} = a_3 \to a_2.$$

General Product of Fuzzy Relations

Definition 2. For an aggregation structure $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box \rangle$, and fuzzy relations $R \in L_1^{X \times Y}$, $S \in L_2^{Y \times Z}$, let a fuzzy relation $R \boxdot S \in L_3^{X \times Z}$ be defined by

$$(R \boxdot S)(x, z) = \bigvee\nolimits_{3\, y \in Y} (R(x, y) \Box S(y, z)).$$

Product $\boxdot$ generalizes both sup-t-norm product ($\circ$) and inf-residuum product ($\lhd$):

- for the setting of Example 1: $R \boxdot S = R \circ S$,
- for the setting of Example 2: $R \boxdot S = R \lhd S$.

Definition 3. For $R \in L_1^{X \times Y}$ and $S \in L_3^{Y \times Z}$, let $R \lhd_{\Box} S \in L_2^{X \times Z}$ and $R \,{}_{\Box}\!\lhd\, S \in L_1^{X \times Z}$ be defined by

$$(R \lhd_{\Box} S)(x, z) = \bigwedge\nolimits_{2\, y \in Y} (R(x, y) \circ_{\Box} S(y, z)),$$
$$(R \,{}_{\Box}\!\lhd\, S)(x, z) = \bigwedge\nolimits_{1\, y \in Y} (R(x, y) \,{}_{\Box}\!\circ\, S(y, z)).$$

Fuzzy Relation Equations

Fuzzy relational equations play an important role in fuzzy set theory and its applications. Namely, it is often the case that the problem in a particular application of fuzzy logic may be transformed to the problem of identification an unknown fuzzy relation.

The problem to determine an unknown fuzzy relation R between universe sets X and Y such that

$$R \boxdot S = T,$$

where S and T are given (known) fuzzy relations between Y and Z, and X and Z, respectively, and $\boxdot$ is an operation of composition of fuzzy relations, is called the problem of fuzzy relational equations. Alternatively, given R and T, the problem is to determine S.

Denotation:

$$U \boxdot S = T \quad \text{and} \quad R \boxdot U = T,$$

where U is the unknown fuzzy relation.

Solvability Criteria

Theorem 4. *Let $R \in L_1^{X \times Y}$, $S \in L_2^{Y \times Z}$, and $T \in L_3^{X \times Z}$ be fuzzy relations. Then*

- $U \boxdot S = T$ has a solution iff $T \,{}_{\Box}\!\lhd\, S^{-1}$ is its solution,
- $R \boxdot U = T$ has a solution iff $R^{-1} \lhd_{\Box} T$ is its solution.

Theorem 5. *If an equation $U \boxdot S = T$ is solvable then the set of all solutions along with $\subseteq_1$ forms a complete join-semilattice with the greatest element*

$$T \,{}_{\Box}\!\lhd\, S^{-1}.$$

If an equation $R \boxdot U = T$ is solvable then the set of all solutions along with $\subseteq_2$ forms a complete join-semilattice with the greatest element

$$R^{-1} \lhd_{\Box} T.$$

Corollary 6. *An equation $U \circ S = T$ is solvable iff $(S \lhd T^{-1})^{-1}$ is its solution. An equation $R \circ U = T$ is solvable iff $R^{-1} \lhd T$ is its solution.*

All Solutions vs. Minimal Solutions

We assume two solutions R', R'' of $U \boxdot S = T$, and fuzzy relation $R \in L_1^{X \times Y}$ such that $R' \subseteq_1 R \subseteq_1 R''$. Since $\boxdot$ is isotone in both arguments, we can write

$$T = R' \boxdot S \subseteq_1 R \boxdot S \subseteq_1 R'' \boxdot S = T,$$

which implies that R is a solution of $U \boxdot S = T$ as well.

In other words, a set of all solutions of $U \boxdot S = T$ w.r.t. $\subseteq_1$ is a convex set. Therefore, we just need to find all minimal solutions.

Minimal Solutions

"Scalar-by-Scalar Equation"

Let $u \in L_1$, $s \in L_2$, $t \in L_3$, define scalar-by-scalar equation:

$$u \Box s = t. \tag{1}$$

Theorem 7. *If equation (1) is solvable then for every its solution $r \in L_1$ it holds $r \in [t \,{}_{\Box}\!\circ\, s, t \overset{\mathrm{op}}{\Box}\!\circ s]$.*

Lemma 8. *Let $\mathbf{L}_1$ be a finite chain. If there is $a_1 \in L_1$ such that $a_1 \Box a_2 \geq_3 a_3$ then $a_1 \overset{\mathrm{op}}{\Box}\!\circ a_3 = a_1 \,{}_{\Box}\!\circ\, a_3$.*

Corollary 9. *Let $\mathbf{L}_1$ be a finite chain. Equation (1) is solvable iff $t \,{}_{\Box}\!\circ\, s$ is its solution.*

"Vector-by-Vector Equation"

Let $(u_j) \in L_1^Y$, $(s_j) \in L_2^Y$, $t \in L_3$, $j \in J = \{1, \dots, n\}$, define vector-by-vector equation:

$$\begin{pmatrix} u_1 \dots u_n \end{pmatrix} \boxdot \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} = t. \tag{2}$$

Theorem 10. *Let $\mathbf{L}_1$ be a finite chain. If there is $j' \in J$ such that $u_{j'} \Box s_{j'} = t$ is solvable then equation (2) has a minimal solution $R = (r_1 \dots r_n)$ such that*

$$r_j = \begin{cases} t \,{}_{\Box}\!\circ\, s_j, & \text{for } j = j', \\ 0_1, & \text{otherwise.} \end{cases}$$

"Vector-by-Matrix Equation"

Let $(u_j) \in L_1^Y$, $(s_{jk}) \in L_2^{Y \times Z}$, $(t_k) \in L_3^Z$, $j \in J = \{1, \dots, n\}$, $k \in K = \{1, \dots, p\}$, and $\mathbf{L}_1$ be a finite chain, define vector-by-matrix equation:

$$\begin{pmatrix} u_1 \dots u_n \end{pmatrix} \boxdot \begin{pmatrix} s_{11} \dots s_{1p} \\ \vdots \quad\quad \vdots \\ s_{n1} \dots s_{np} \end{pmatrix} = \begin{pmatrix} t_1 \dots t_p \end{pmatrix}. \tag{3}$$

Equation (3) can be rewritten using a table $\mathfrak{T}$ of dimension $(n+1) \times p$ (meaning, the last row is $\vee_3$ of the rows above):

$u_1 \Box s_{11}$	$\dots$	$u_1 \Box s_{1p}$
$\vdots$		$\vdots$
$u_n \Box s_{n1}$	$\dots$	$u_n \Box s_{np}$
t_1	$\dots$	t_p

Now, see what happen to the table $\mathfrak{T}$ when we take $R = (r_1 \dots r_n) = T \,{}_{\Box}\!\lhd\, S^{-1}$ (the greatest solution) as a solution of (3).

For every $j \in J$ there must be $k' \in K$ such that

$$r_j = \bigwedge\nolimits_{1\, k \in K} (t_k \,{}_{\Box}\!\circ\, s_{jk}) = t_{k'} \,{}_{\Box}\!\circ\, s_{jk'}.$$

By K_j we denote the set of all indices $k' \in K$ such that $u_j \Box s_{jk'} = t_{k'}$ is solvable and $r_j = t_{k'} \,{}_{\Box}\!\circ\, s_{jk'}$.

Minimal Solutions

Important assertions:

- for each $k' \in K_j$, $r_j = t_{k'} \,{}_{\Box}\!\circ\, s_{jk'}$ is the only solution of $u_j \Box s_{jk'} = t_{k'}$
- for each $k'' \in K \setminus K_j$ we have $r_j \Box s_{jk''} <_3 t_{k''}$.

Define a binary table $\mathfrak{B}$ of the dimension $(n+1) \times p$:

$$\mathfrak{B}_{jk} = \begin{cases} 1, & \text{if } \mathfrak{T}_{jk} = t_k, \\ 0, & \text{if } \mathfrak{T}_{jk} <_3 t_k. \end{cases}$$

Obviously, the last row of the table $\mathfrak{B}$ is filled by ones.

Definition 11. $J_{\mathrm{cov}} \subseteq J$ is a covering of the last row of $\mathfrak{B}$ if $\max_{j \in J_{\mathrm{cov}}} \mathfrak{B}_{jk} = 1$ for all $k \in K$. Covering $J_{\mathrm{cov}} \subseteq J$ is a minimal one if there is no covering J'_{cov} such that $J'_{\mathrm{cov}} \subseteq J_{\mathrm{cov}}$.

Example 3. Assume an equation (3) with table $\mathfrak{T}$:

$r_1 \Box s_{11} = t_1$	$r_1 \Box s_{12} = t_2$	$r_1 \Box s_{13} <_3 t_3$	$r_1 \Box s_{14} <_3 t_4$
$r_2 \Box s_{21} <_3 t_1$	$r_2 \Box s_{22} <_3 t_2$	$r_2 \Box s_{23} = t_3$	$r_2 \Box s_{24} = t_4$
$r_3 \Box s_{31} = t_1$	$r_3 \Box s_{32} <_3 t_2$	$r_3 \Box s_{33} <_3 t_3$	$r_3 \Box s_{34} = t_4$
$r_4 \Box s_{41} <_3 t_1$	$r_4 \Box s_{42} <_3 t_2$	$r_4 \Box s_{43} = t_3$	$r_4 \Box s_{44} <_3 t_4$
t_1	t_2	t_3	t_4

Table $\mathfrak{B}$ can be easily derived from $\mathfrak{T}$:

1	1	0	0
0	0	1	1
1	0	0	1
0	0	1	0
1	1	1	1

There exist several coverings, but just two of them are the minimal ones: $\{1, 2\}$, $\{1, 3, 4\}$.

Theorem 12. *Let (3) be an equation with $R = (r_1 \dots r_n)$ being the greatest solution. Every minimal solution $M = (m_1 \dots m_n)$ of (3) is in the form:*

$$m_j = \begin{cases} r_j, & \text{for } j \in J_{\mathrm{cov}}, \\ 0_1, & \text{otherwise}, \end{cases}$$

where J_{cov} is a minimal covering of the last row of the corresponding table $\mathfrak{B}$.

Minimal solutions of $U \circ S = T$ (sup-t-norm equation) and $U \lhd S = T$ (inf-residuum equation) can be described using direct consequencies of the previous results.

Future Research

- developing algorithms, efficient computation of all solutions (removing duplicities), complexity issues

References

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