Ivan Chajda, Miroslav Kolařík, Jan Krňávek: Pseudo basic algebras

Faculty of Science, Palacky University, Olomouc (17. listopadu 12, CZ–77146 Olomouc, Czech Republic)

Abstract and Motivation

Abstract

Since basic algebras are equivalent to bounded lattices with sectional antitone involutions, it motivated us to study an algebraic counterpart of semilattices with sectional switching involutions. These algebras are called pseudo basic algebras. They are determined by four independent identities and hence the class of these algebras forms a variety. Several basic properties of these algebras are presented and a particular interest is devoted to pseudo basic algebras whose main involution is even antitone (so-called strict pseudo basic algebras) and to those whose binary operation is commutative. Congruence properties of the varieties of these algebras are investigated.

Motivation

The concept of basic algebra was introduced by the first author, R. Halaš and J. Kühr as a common generalization of an MV-algebra and an orthomodular lattice. Remember that MV-algebras serve as an algebraic axiomatization of the so-called Łukasiewicz many-valued logics and orthomodular lattices form an algebraic counterpart of the logic of quantum mechanics. Hence, basic algebras form a common algebraic axiomatization of both logics mentioned above.
An interesting connection of commutative basic algebras and certain non-associative fuzzy logics is pointed out by M. Botur and R. Halaš. These logics play an important role in so-called expert systems. From the algebraic point of view, the logical connective implication is expressed as a term operation $x \to y = \neg x \oplus y$. Applying the alter ego of basic algebras, i.e. lattices with sectional antitone involutions, implication can be equivalently defined by $x \to y = (x \vee y)^y$. Since this expression does not contain the lattice operation $\wedge$, it can be used also for semilattices with sectional involutions which is just our case. At the first glance, one can see that antitonicity does not play a role in this definition. Hence, we can introduce a logic (based on implication and, possibly, negation and constants) which can be algebraically axiomatized via this algebra. In what follows, we will call this algebra as pseudo basic algebra and we will search for its axiomatization and important algebraic properties.

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Basic concepts

Consider a section $[a,1]$ of an ordered set with greatest element 1. A mapping $x \mapsto x^a$ of $[a,1]$ into itself is called a **sectional switching involution** if $x^{aa} = x$ for each $x \in [a,1]$ and $a^a = 1, 1^a = a$. In general, we do not ask that this involution should be antitone; it only switches the endpoints of the section.
Now, we can consider a bounded lattice with sectional switching involutions $\mathcal{L} = (L; \vee, \wedge, (^a)_{a\in L}, 0, 1)$ and study what an algebra can be obtained by using a similar construction as that for basic algebras. For our purposes, we will consider only a semilattice since the operation meet is not applied in the construction of the operations of the new algebra.

THEOREM 1. Let $\mathcal{S} = (S; \vee, (^a)_{a\in S}, 0, 1)$ be a bounded semilattice with sectional switching involutions. Define $\neg x = x^0$ and $x \oplus y = (x^0 \vee y)^y$. Then the algebra $\mathcal{A}(\mathcal{S}) = (S; \oplus, \neg, 0)$ assigned to $\mathcal{S}$ satisfies the following identities:

(1) $\neg x \oplus x = 1$

(2) $x \oplus 0 = x$

(3) $\neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y$

(4) $\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z = \neg(\neg(\neg(\neg y \oplus z) \oplus z) \oplus x) \oplus x$.

If, moreover, the involution $x \mapsto x^0$ is antitone, then $\mathcal{A}(\mathcal{S})$ satisfies the identity

(A) $(\neg(x \oplus y) \oplus y) \oplus x = 1$.

REMARK 1.

(a) Every bounded semilattice $\mathcal{S} = (S; \vee, 0, 1)$ can be considered as a semilattice with sectional switching involutions. Namely, for each $a \in S$ one can define a switching involution on $[a,1]$ as follows: $a^a = 1, 1^a = a$ and $x^a = x$ for each $x \in [a,1], a \neq x \neq 1$.

(b) If the involution $x \mapsto x^0$ is antitone, then $\mathcal{S} = (S; \vee, 0, 1)$ is in fact a lattice due to the DeMorgan laws because
$$x \wedge y = (x^0 \vee y^0)^0.$$

(c) There exist bounded semilattices with sectional switching involutions which are not lattices.

Assigned algebras

We have shown that to every bounded semilattice $\mathcal{S}$ with sectional switching involutions we can assign an algebra $\mathcal{A}(\mathcal{S}) = (S; \oplus, \neg, 0)$ satisfying (1)–(4). We are going to show that algebras satisfying (1)–(4) are interesting for their own sake.

DEFINITION 1. An algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2,1,0)$ satisfying the identities (1)–(4) will be called a **pseudo basic algebra**. If, moreover, $\mathcal{A}$ satisfies also the identity (A), it will be called a **strict pseudo basic algebra**.

Since both pseudo basic algebras and strict pseudo basic algebras are determined by identities, their classes are in fact varieties.

THEOREM 2. The axioms (1)–(4) are independent.

Our next task is to show that also conversely, every pseudo basic algebra can be organized into a bounded semilattice with sectional switching involutions.

THEOREM 3. Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a pseudo basic algebra. Define $1 = \neg 0$, $x \vee y = \neg(\neg x \oplus y) \oplus y$ and for any $a \in A$, let $x^a = \neg x \oplus a$. Then

(a) $(A; \vee)$ is a join-semilattice with least element 0 and greatest element 1

(b) $x \leq y$ if and only if $\neg x \oplus y = 1$ is the induced order of the semilattice $(A; \vee)$

(c) for each $a \in A$ and $x \in [a,1]$, the mapping $x \mapsto x^a = \neg x \oplus a$ is a sectional switching involution on the section $[a,1]$.

If, moreover, $\mathcal{A}$ is a strict pseudo basic algebra, then $(A; \vee)$ is a lattice where $x \wedge y = \neg(\neg x \vee \neg y)$.

We can show that the assignment between pseudo basic algebras and bounded semilattices with sectional switching involutions is a one-to-one correspondence.

THEOREM 4. Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a pseudo basic algebra, $\mathcal{S}(\mathcal{A})$ its assigned semilattice with sectional switching involutions. Then $\mathcal{A}(\mathcal{S}(\mathcal{A})) = \mathcal{A}$.
Let $\mathcal{S} = (S; \vee, (^a)_{a\in S}, 0, 1)$ be a bounded semilattice with sectional switching involutions, $\mathcal{A}(\mathcal{S})$ its assigned pseudo basic algebra. Then $\mathcal{S}(\mathcal{A}(\mathcal{S})) = \mathcal{S}$.

Strict and commutative PBA

THEOREM 5. Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a strict pseudo basic algebra. Then $\neg$ is a complementation in the induced lattice $(A; \vee, \wedge, 0, 1)$ if and only if $\mathcal{A}$ satisfies the identity $x \oplus x = x$.

A pseudo basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is called **commutative** if it satisfies the identity $x \oplus y = y \oplus x$ and $\mathcal{A}$ is called **associative** if it satisfies the identity $x \oplus (y \oplus z) = (x \oplus y) \oplus z$. An interesting connection is given by the following.

THEOREM 6.

(a) Every commutative pseudo basic algebra is strict.

(b) A pseudo basic algebra is an MV-algebra if and only if it is associative.

It was proved by M. Botur and R. Halaš that every finite commutative basic algebra is in fact an MV-algebra. Hence, it is a natural question if there really exist commutative pseudo basic algebras which are not basic algebras. The answer is positive also for a finite pseudo basic algebra.

Congruence properties and section algebras

THEOREM 7. The variety of pseudo basic algebras is weakly regular. The variety of strict pseudo basic algebras is arithmetical and congruence regular.

As shown above, pseudo basic algebras are equivalent to bounded join-semilattices with sectional switching involutions. However, every section is a semilattice, i.e. it is again a bounded semilattice with sectional switching involutions. Hence, it can be converted into a pseudo basic algebra. How to organize its operations is shown in the following.

THEOREM 8. Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a pseudo basic algebra, let $\leq$ be its induced order and $p \in A$. The section $[p,1]$ can be organized into a pseudo basic algebra $([p,1]; \oplus_p, \neg_p, p)$ as follows:
$$\neg_p x = \neg x \oplus p \quad \text{and} \quad x \oplus_p y = \neg(\neg x \oplus p) \oplus y$$
for $x, y \in [p,1]$.