R. Belohlavek, T. Kuhr, V. Vychodil: Confluence of Fuzzy Relations

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Introduction

Abstract rewriting systems deal with the idea of substituting elements by other elements which are indistinguishable (from certain point of view) from the original ones. In particular, in term rewriting systems, complex terms are substituted by simpler terms that have the same meaning. As an example, $x + x + x$ can be replaced by an equivalent but simpler term $3 \times x$ considering the usual interpreation of $+$ and $\times$.

Our research is motivated by the fact that in many cases the notion of substitutability is inherently fuzzy rather than crisp. That is, instead of substituting equal elements for equal ones, one may wish to substitute similar elements for similar ones. For instance, one often substitutes an option y for option x whenever y is much cheaper than x and y does the job of x sufficiently well.

These results were already published in the following papers:

:: Belohlavek, R., Kuhr, T. and Vychodil, V.: Confluence and termination of fuzzy relations. Information Sciences.

:: Kuhr, T., Vychodil, V.: Similarity issues of confluence of fuzzy relations. International Journal of General Systems.

Structures of truth degrees

As structures of truth degrees, we use complete residuated lattices which are algebras $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that

:: $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice,

:: $\langle L, \otimes, 1\rangle$ is a commutative monoid,

:: $\otimes$ and $\rightarrow$ satisfies $a \otimes b \le c$ iff $a \le b \rightarrow c$, for all $a, b, c \in L$.

The Łukasiewicz structure of truth degrees is a complete residuated lattice on the unit interval with conjunction and residuum being:

$$a \otimes b = \max(0, a + b - 1) \text{ and } a \rightarrow b = \min(1, 1 - a + b).$$

The two-valued Boolean algebra $\mathbf{2}$ is also a particular case of a complete residuated lattice.
We recall that $a \in L$ is called idempotent if $a \otimes a = a$.

Fuzzy sets and relations

An $\mathbf{L}$-set in a universe set X is any map $A\colon X \to L$.
A binary $\mathbf{L}$-relation on X is a map $R\colon X \times X \to L$.
The $\circ$-composition of binary $\mathbf{L}$-relations R_1, R_2 on X, is defined by

$$x \rightharpoonup_1 \circ \rightharpoonup_2 y = \textstyle\bigvee_{z \in X} (x \rightharpoonup_1 z \otimes z \rightharpoonup_2 y).$$

For $\mathbf{L}$-sets A and B in X we define:

$$S(A, B) = \textstyle\bigwedge_{x \in X} \big(A(x) \rightarrow B(x)\big),$$
$$E(A, B) = \textstyle\bigwedge_{x \in X} \big(A(x) \leftrightarrow B(x)\big).$$

$S(A, B)$ is called a degree of subsethood of A in B;
$E(A, B)$ is called a degree of equality of A and B.

Confluence of fuzzy relations

In the following, let $\mathbf{L}$ be a complete residuated lattice and $\rightharpoonup$ be a binary $\mathbf{L}$-relation on a universe set X.

Definition 1. We define an $\mathbf{L}$-relation $\rightharpoonup^*$ on X by

$$x \rightharpoonup^* y = \textstyle\bigvee_{\langle z_1, z_2, \ldots, z_k\rangle \in X^*} (x \rightharpoonup z_1 \otimes \cdots \otimes z_k \rightharpoonup y),$$

where $X^* = \bigcup_{n \in \mathbb{N}_0} X^n$, for $x \neq y$ and $x \rightharpoonup^* y = 1$ otherwise.
The $\mathbf{L}$-relation $\rightharpoonup^*$ is called reducibility induced by $\rightharpoonup$.

Remark 2. The definition of reducibility can be alternatively written as $\rightharpoonup^* = \bigcup_{n=0}^{\infty} \rightharpoonup^n$, where $\rightharpoonup^n$ is defined by $\rightharpoonup^n = \rightharpoonup \circ \rightharpoonup^{n-1}$ $(n \ge 1)$, $\rightharpoonup^0$ being the identity $\mathbf{L}$-relation.
As in the ordinary case, $\rightharpoonup^*$ is the least reflexive and transitive fuzzy relation containing $\rightharpoonup$.

Definition 3. An $\mathbf{L}$-relation $\downarrow$ defined by

$$x \downarrow y = \textstyle\bigvee_{z \in X} (x \rightharpoonup^* z \otimes y \rightharpoonup^* z).$$

is called convergence.

Convertibility $\mathbf{L}$-relation $\rightleftharpoons^*$ is defined as a reflexive, symmetric and transitive closure of $\rightharpoonup$.

The degree $\mathrm{CR}(\rightharpoonup)$ to which $\rightharpoonup$ has the Church-Rosser property is defined by $\mathrm{CR}(\rightharpoonup) = S(\rightleftharpoons^*, \downarrow)$.

Remark 4. Directly from definition, $\downarrow = \rightharpoonup^* \circ \leftharpoondown^*$, where $\leftharpoondown$ denotes an inverse $\mathbf{L}$-relation to $\rightharpoonup$.

Note that if $\mathbf{L} = \mathbf{2}$ then

:: $x \downarrow y = 1$ iff x and y are convergent in the usual sense,

:: $x \rightleftharpoons^* y = 1$ iff x and y are ordinarily convertible and

:: $\mathrm{CR}(\rightharpoonup) = 1$ iff $\rightharpoonup$ has the (ordinary) Church-Rosser property.

Theorem 5. $\mathrm{CR}(\rightharpoonup) = E(\rightleftharpoons^*, \downarrow)$.

Remark 6. Theorem 5 generalizes the classical theorem which is well known in the theory of abstract rewriting systems in the following way. If we let $\mathbf{L} = \mathbf{2}$, Theorem 5 is equivalent to saying that $\mathrm{CR}(\rightharpoonup) = 1$ iff $\rightleftharpoons^* = \downarrow$, i.e. $\rightharpoonup$ has the Church-Rosser property iff the convertibility and convergence relations coincide.

Definition 7. We define divergence by

$$x \uparrow y = \textstyle\bigvee_{z \in X} (z \rightharpoonup^* x \otimes z \rightharpoonup^* y).$$

The degree $\mathrm{CFL}(\rightharpoonup)$ to which $\rightharpoonup$ is confluent is defined by $\mathrm{CFL}(\rightharpoonup) = S(\uparrow, \downarrow)$.

Remark 8. Analogously as in case of convergence, we have $\uparrow = \leftharpoondown^* \circ \rightharpoonup^*$.
If $\mathbf{L} = \mathbf{2}$, the notions of divergence and confluence fully correspond to the classical properties.

Theorem 9. *If* $\mathrm{CFL}(\rightharpoonup)$ *is an idempotent element of* $\mathbf{L}$ *then* $\mathrm{CR}(\rightharpoonup) = \mathrm{CFL}(\rightharpoonup)$.

Similarity issues

In this part, we investigate the properties (reducibility, convergence, divergence and confluence) of 2 binary $\mathbf{L}$-relations $\rightharpoonup_1$ and $\rightharpoonup_2$ on X.

Monotony of reducibility degrees

First, we have studied properties of the operator * which associates to each $\rightharpoonup$ the reducibility $\rightharpoonup^*$. Directly from its definition, we can observe that * is monotone in the usual sense, i.e., $x \rightharpoonup_1 y \le x \rightharpoonup_2 y$ implies $x \rightharpoonup_1^* y \le x \rightharpoonup_2^* y$.
Hence, $\rightharpoonup_1 \subseteq \rightharpoonup_2$ implies $\rightharpoonup_1^* \subseteq \rightharpoonup_2^*$ for any $\rightharpoonup_1, \rightharpoonup_2$. On the other hand, notice that a common stronger form of monotony which appears in fuzzy setting, i.e. $S(\rightharpoonup_1, \rightharpoonup_2) \le S(\rightharpoonup_1^*, \rightharpoonup_2^*)$, does not hold in general (see Figure 1).

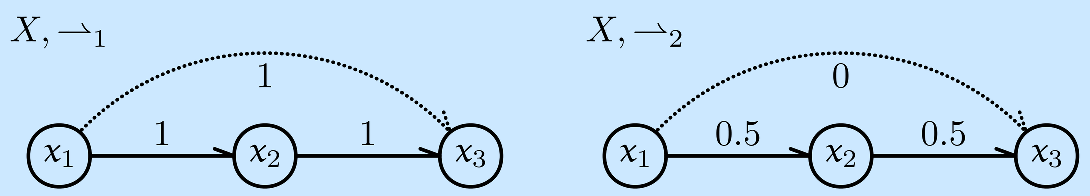

Figure 1: Counterexample to the common stronger form of monotony, $\mathbf{L}$ is the Łukasiewicz structure.

However, we can prove the following stronger form of monotony:

Theorem 10. $\bigwedge_{n \in \mathbb{N}_0} S(\rightharpoonup_1, \rightharpoonup_2)^n \le S((\rightharpoonup_1)^*, (\rightharpoonup_2)^*)$.

Sensitivity of confluence

Now, we show that for similar $\mathbf{L}$-relations $\rightharpoonup_1$ and $\rightharpoonup_2$ we obtain more or less similar reducibility, convergence and divergence $\mathbf{L}$-relations. The degrees of confluence are also "somehow similar".

Theorem 11.

$$\textstyle\bigwedge_{n \in \mathbb{N}_0} E(\rightharpoonup_1, \rightharpoonup_2)^n \le E(\rightharpoonup_1^*, \rightharpoonup_2^*),$$
$$\big(\textstyle\bigwedge_{n \in \mathbb{N}_0} E(\rightharpoonup_1, \rightharpoonup_2)^n\big)^2 \le E(\uparrow_1, \uparrow_2),$$
$$\big(\textstyle\bigwedge_{n \in \mathbb{N}_0} E(\rightharpoonup_1, \rightharpoonup_2)^n\big)^2 \le E(\downarrow_1, \downarrow_2),$$
$$\big(\textstyle\bigwedge_{n \in \mathbb{N}_0} E(\rightharpoonup_1, \rightharpoonup_2)^n\big)^4 \le \mathrm{CFL}(\rightharpoonup_1) \leftrightarrow \mathrm{CFL}(\rightharpoonup_2).$$

There are important cases in which $\rightharpoonup_1$ and $\rightharpoonup_2$ are similar to a degree which is an idempotent element of $\mathbf{L}$. In these cases, the results of Theorem 11 can be even simplified.

Corollary 12. *If* $E(\rightharpoonup_1, \rightharpoonup_2)$ *is an idempotent element of* $\mathbf{L}$, *then*

$$E(\rightharpoonup_1, \rightharpoonup_2) \le E(\rightharpoonup_1^*, \rightharpoonup_2^*),$$
$$E(\rightharpoonup_1, \rightharpoonup_2) \le E(\uparrow_1, \uparrow_2),$$
$$E(\rightharpoonup_1, \rightharpoonup_2) \le E(\downarrow_1, \downarrow_2),$$
$$E(\rightharpoonup_1, \rightharpoonup_2) \le \mathrm{CFL}(\rightharpoonup_1) \leftrightarrow \mathrm{CFL}(\rightharpoonup_2).$$

Example of an confluent fuzzy relation

Example 13. The Figure 2 contains a diagram of an $\mathbf{L}$-relation $\rightharpoonup$ on a set X of selected colors. The degree $x \rightharpoonup y$ to which x is $\rightharpoonup$-related with y can be interpreted as a degree to which "color y can be substituted for color x". The degrees to which colors can be substituted by each other can be determined based on their wavelength and based on the human perception of "color similarity", which is a complex neuro-chemical process with a psychological feedback that varies from person to person. The illustration in Figure 2 show how the authors perceive a substitutability of lighter color for darker ones. The truth values were taken from an eleven-valued Łukasiewicz algebra. It can be shown that $\rightharpoonup$ is an $\mathbf{L}$-relation with a high degree of confluence. Namely, $\mathrm{CFL}(\rightharpoonup) = 0.9$.

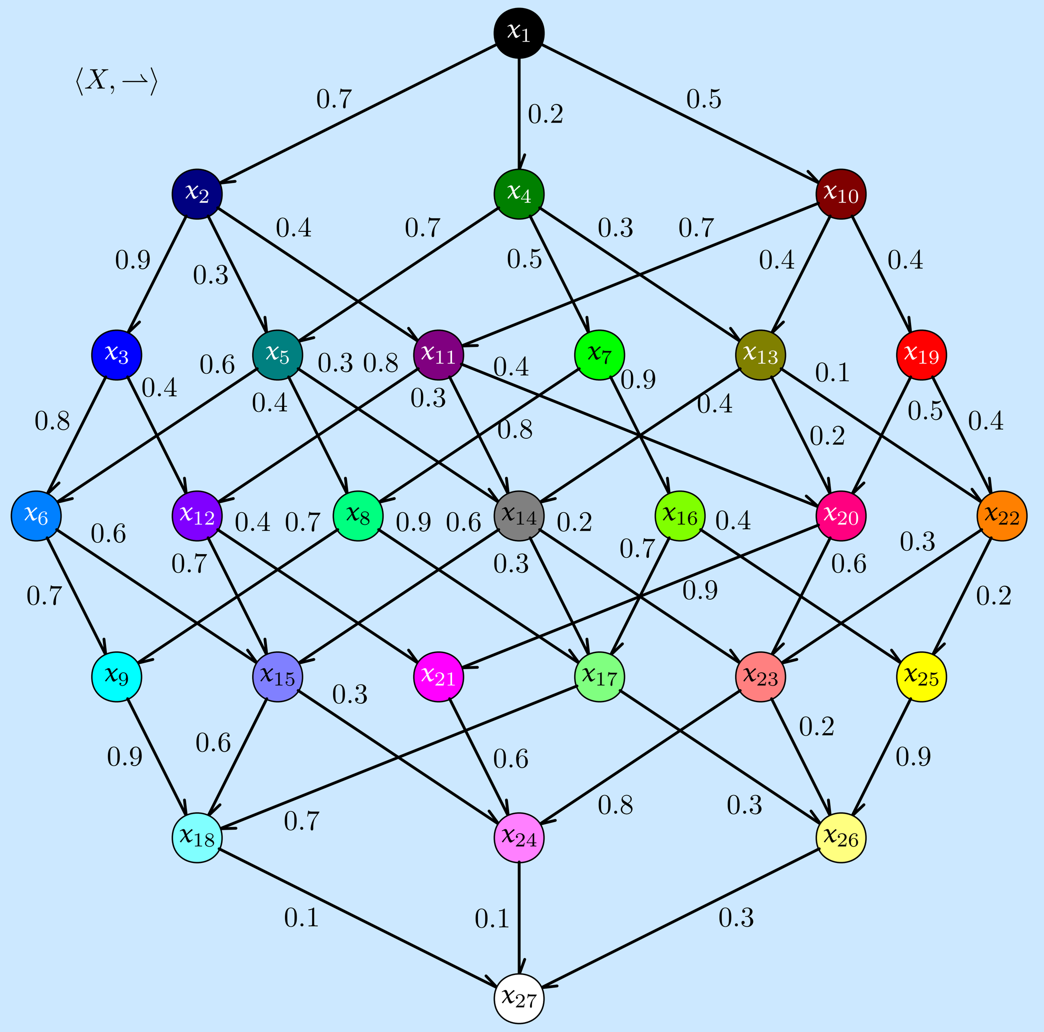

Figure 2: Fuzzy relation "color y can be substituted for x".

Conclusions

We have defined reducibility, convergence, divergence, convertibility, Church-Rosser property and confluence of fuzzy relations and investigated their graded properties. We have shown that these notions have analogous properties and mutual relationship as in the ordinary case. We have also studied basic similarity issues of these properties of fuzzy relations, which results in a collection of formulas providing lower estimations of their degrees.

Future research

Our future research will be focused on the notions related to confluence of fuzzy relations defined on similarity and metric spaces.

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