Jan Laštovička: Example of Incomplete Concept Lattice

Department of Computer Science, Palacky University, Olomouc (17. listopadu 12, CZ–77146 Olomouc, Czech Republic)

Introduction

In practice, we often have to deal with incomplete data. Concept lattices of incomplete data have been studied in [1] and [2]. This poster gives one example of concept lattices of incomplete data.

Formal Concept Analysis

A formal context is a triple $\langle X, Y, I\rangle$ where X is a set of objects, Y a set of attributes and $I \subseteq X \times Y$. For $\langle x, y\rangle \in I$ it is said "The object x has the attribute y".
For subsets $A \subseteq X$ and $B \subseteq Y$ we set

$$A^{\uparrow} = \{y \in Y \mid \text{for each } x \in A \text{ it holds } \langle x, y\rangle \in I\},$$
$$B^{\downarrow} = \{x \in X \mid \text{for each } y \in B \text{ it holds } \langle x, y\rangle \in I\}.$$

If $A^{\uparrow} = B$ and $B^{\downarrow} = A$, then the pair $\langle A, B\rangle$ is called a formal concept of $\langle X, Y, I\rangle$. The set A is called the extent, B the intent of $\langle A, B\rangle$.
The set of all formal concepts is denoted by $\mathcal{B}(X, Y, I)$.
A partial order $\leq$ on the set $\mathcal{B}(X, Y, I)$ is defined by

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \quad \text{iff} \quad A_1 \subseteq A_2 \quad (\text{iff } B_2 \subseteq B_1).$$

$\mathcal{B}(X, Y, I)$ together with $\leq$ is a complete lattice and is called the concept lattice of $\langle X, Y, I\rangle$.

Formal Concept Analysis in Fuzzy Setting

Let $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ be a complete residuated lattice.
An $\mathbf{L}$-set A in universe X is a mapping $A\colon X \to L$.
The set of all $\mathbf{L}$-sets in universe X is denoted by L^X.
For two $\mathbf{L}$-sets $A, B \in L^X$ we say that A is a subset of B and write $A \subseteq B$, if for each $x \in X$ it holds $A(x) \leq B(x)$.
A formal $\mathbf{L}$-context is a triple $\langle X, Y, I\rangle$, where X and Y are sets and $I\colon X \times Y \to L$. For $\mathbf{L}$-sets $A \in L^X$ and $B \in L^Y$ we set

$$A^{\uparrow}(y) = \bigwedge_{x \in X} A(x) \rightarrow I(x, y),$$
$$B^{\downarrow}(x) = \bigwedge_{y \in Y} B(y) \rightarrow I(x, y).$$

An $\mathbf{L}$-concept is a pair $\langle A, B\rangle \in L^X \times L^Y$ such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$. The $\mathbf{L}$-set A is called the extent, B the intent of $\langle A, B\rangle$.
An $\mathbf{L}$-concept $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ is called crisply generated, if there is a crisp set $B_0 \subseteq Y$ such that $A = (B_0)^{\downarrow}$.
The set of all crisply generated $\mathbf{L}$-concepts is denoted by $\mathcal{B}_c(X, Y, I)$.
A partial order $\leq$ on the set $\mathcal{B}_c(X, Y, I)$ is defined by

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \quad \text{iff} \quad A_1 \subseteq A_2 \quad (\text{iff} \quad B_2 \subseteq B_1)$$

$\mathcal{B}_c(X, Y, I)$ together with $\leq$ is a complete lattice.

Theory

Incomplete Contexts

Let $U = \{u_1, \dots, u_k\}$ be a set of variables, $\mathbf{L} = \langle L, \wedge, \vee, ', 0, 1\rangle$ be a finite Boolean algebra (BA), $\iota\colon U \to L$, such that $\mathbf{L}$ is generated by $\iota(U)$. $\mathbf{L}$ together with ι is called a Boolean algebra with variables $u_1, \dots, u_k$. Elements of $\mathbf{L}$ can be viewed as terms, constructed from variables by means of operations of a Boolean algebra.

Mappings $v\colon U \to \mathbf{2}$ are called assignments. Since $\mathbf{L}$ is generated by $\iota(U)$, then for each assignment v there exists at most one homomorphism $\bar{v}\colon \mathbf{L} \to \mathbf{2}$, such that $v = \bar{v} \circ \iota$. If this homomorphism exists, then the assignment v is called admissible.

The class of Boolean algebras can be considered a subclass of the class of residuated lattices.

An incomplete $\mathbf{L}$-context is an $\mathbf{L}$-context $\langle X, Y, I\rangle$ such that $I(X \times Y) \subseteq U \cup \{0, 1\}$.

An ordinary formal context $\langle X, Y, J\rangle$ is a completion of $\langle X, Y, I\rangle$, if $J = \bar{v} \circ I$ for an admissible assignment $v\colon U \to \mathbf{2}$.

Construction of BA with Variables

The following holds for each subset $V \subseteq \mathbf{2}^U$ (known dependencies between variables).

- (Generality) Let $\mathbf{L} = \mathbf{2}^V$ and a mapping $\iota\colon U \to L$ be defined by $(\iota(u))(v) = v(u)$. Then for each $v \in V$ there is exactly one homomorphism $\bar{v}\colon \mathbf{L} \to \mathbf{2}$ such that $v = \bar{v} \circ \iota$.
- (Efficiency) Let $\mathbf{L}'$ be a residuated lattice and $\iota'\colon U \to L'$ a mapping such that for each $v \in V$ there is a homomorphism $\bar{v}'\colon \mathbf{L}' \to \mathbf{2}$ satisfying $v = \bar{v}' \circ \iota'$. Then there exists a surjective homomorphism of residuated lattices $s\colon \mathbf{L}' \to \mathbf{L}$ such that for each $v \in V$ it holds $\bar{v}' = \bar{v} \circ s$.

Incomplete Concept Lattices

Let $\langle X, Y, I\rangle$ be an incomplete $\mathbf{L}$-context and v be an admissible assignment.

Then

- $\langle X, Y, \bar{v} \circ I\rangle$ is a completion of $\langle X, Y, I\rangle$,
- $\mathcal{B}(X, Y, \bar{v} \circ I)$ is a classical concept lattice,
- mapping $\bar{v}^{\mathcal{B}_c(X,Y,I)}\colon \mathcal{B}_c(X, Y, I) \to \mathcal{B}(X, Y, \bar{v} \circ I)$ defined by $\bar{v}^{\mathcal{B}_c(X,Y,I)}(\langle A, B\rangle) = \langle \bar{v} \circ A, \bar{v} \circ B\rangle$ is a surjective, $\bigwedge$-preserving complete homomorphism.

Example

The Festival Context

A group of people plans to visit some music festivals, we partially know their plans. We also know which festivals they have visited last summer. Considered festivals are Open Air Festival Trutnov (oaft), Colours of Ostrava (co), Rock for People (rp), Mighty Sounds (ms), Brutal Assault (ba).

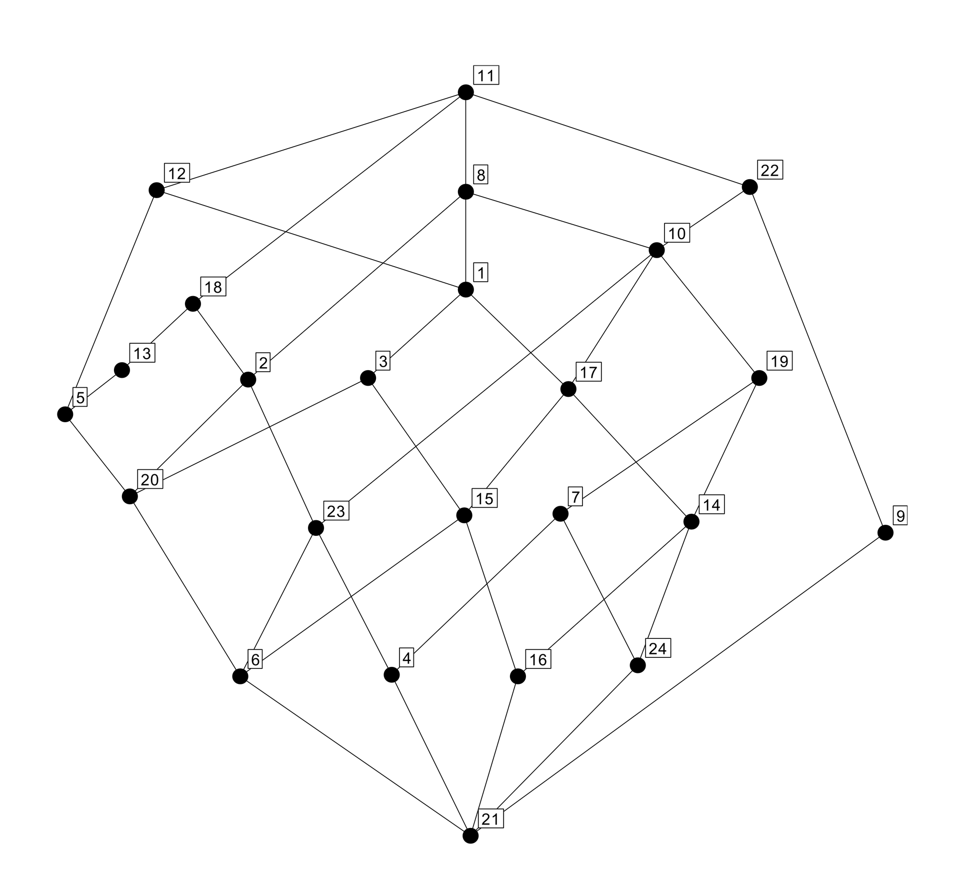
Let $U = \{u_1, u_2, u_3, u_4\}$ be a set of variables, $V = \{v \in \mathbf{2}^U \mid v(u_1) \neq v(u_2) \text{ and } v(u_3) \leq v(u_4)\}$, where dependency $v(u_1) \neq v(u_2)$ means that the hippie woman will visit either CO or MS (she has not much money) and dependency $v(u_3) \leq v(u_4)$ means that the intellectual will visit OAFT if the rocker will do (they are friends).
Future, let $\mathbf{L} = \mathbf{2}^V$ be a Boolean algebra with variables u_1, u_2, u_3, u_4. Suppose: $U \subseteq L$, ι is the inclusion $U \to L$.

Let $\langle X, Y, I\rangle$ be an incomplete $\mathbf{L}$-context (festival context):

	past					future				
	oaft1	co1	rp1	ms1	ba1	oaft2	co2	rp2	ms2	ba2
punk man (p)	X		X	X		X			X	
heavy metal fan (m)					X	X				X
hippie woman (h)	X	X		X		X	u_1		u_2	
emo boy (e)			X					X		
intellectual (i)	X	X				u_4	X			
rocker (r)	X	X	X			u_3	X	X		
shampoo guy (s)		X	X					X		

The Festival Concept Lattice

Structure of $\mathcal{B}_c(X, Y, I)$ of the festival context:



Example

Concepts of the Festival Context

Let $A \in L^X$ be a $\mathbf{L}$-set. If for all $x \in X$ distinct from $x_1, x_2, \dots, x_n$ we have $A(x) = 0$, we write $\{{}^{A(x_1)}/x_1, {}^{A(x_2)}/x_1, \dots, {}^{A(x_n)}/x_n\}$. We write x instead of ${}^{1}/x$.

Crisply generated $\mathbf{L}$-concepts of the festival context:

number	extent	intent
1	$\{h, i, r\}$	$\{\text{oaft1}, \text{co1}, {}^{u_3}/\text{oaft2}, {}^{u_2'}/\text{co2}\}$
2	$\{p, r\}$	$\{\text{oaft1}, \text{rp1}, {}^{u_3}/\text{oaft2}\}$
3	$\{{}^{u_2'}/h, i, r\}$	$\{\text{oaft1}, \text{co1}, {}^{u_3}/\text{oaft2}, \text{co2}\}$
4	$\{p\}$	$\{\text{oaft1}, \text{rp1}, \text{ms1}, \text{oaft2}, \text{ms2}\}$
5	$\{r, s\}$	$\{\text{co1}, \text{rp1}, \text{rp2}\}$
6	$\{{}^{u_3}/r\}$	$\{\text{oaft1}, \text{co1}, \text{rp1}, {}^{u_3'}/\text{ms1}, {}^{u_3'}/\text{ba1}, \text{oaft2}, \text{co2}, \text{rp2}, {}^{u_3'}/\text{ms2}, {}^{u_3'}/\text{ba2}\}$
7	$\{p, {}^{u_2}/h\}$	$\{\text{oaft1}, {}^{u_2'}/\text{rp1}, \text{ms1}, \text{oaft2}, \text{ms2}\}$
8	$\{p, h, i, r\}$	$\{\text{oaft1}, {}^{u_3}/\text{oaft2}\}$
9	$\{m\}$	$\{\text{ba1}, \text{oaft2}, \text{ba2}\}$
10	$\{p, h, {}^{u_4}/i, {}^{u_3}/r\}$	$\{\text{oaft1}, {}^{u_4'}/\text{ms1}, \text{oaft2}, {}^{u_2 \wedge u_4'}/\text{ms2}\}$
11	$\{p, m, h, e, i, r, s\}$	$\emptyset$
12	$\{h, i, r, s\}$	$\{\text{co1}\}$
13	$\{e, r, s\}$	$\{\text{rp1}, \text{rp2}\}$
14	$\{h\}$	$\{\text{oaft1}, \text{co1}, \text{ms1}, \text{oaft2}, {}^{u_2'}/\text{co2}, {}^{u_2}/\text{ms2}\}$
15	$\{{}^{u_2'}/h, {}^{u_4}/i, {}^{u_3}/r\}$	$\{\text{oaft1}, \text{co1}, {}^{u_2 \wedge u_4'}/\text{rp1}, {}^{u_4'}/\text{ms1}, {}^{u_2 \wedge u_4'}/\text{ba1}, \text{oaft2}, \text{co2}, {}^{u_2 \wedge u_4'}/\text{rp2}, {}^{u_2 \wedge u_4'}/\text{ms2}, {}^{u_2 \wedge u_4'}/\text{ba2}\}$
16	$\{{}^{u_2'}/h\}$	$\{\text{oaft1}, \text{co1}, {}^{u_2}/\text{rp1}, \text{ms1}, {}^{u_2}/\text{ba1}, \text{oaft2}, \text{co2}, {}^{u_2}/\text{rp2}, {}^{u_2}/\text{ms2}, {}^{u_2}/\text{ba2}\}$
17	$\{h, {}^{u_4}/i, {}^{u_3}/r\}$	$\{\text{oaft1}, \text{co1}, {}^{u_4'}/\text{ms1}, \text{oaft2}, {}^{u_2'}/\text{co2}, {}^{u_2 \wedge u_4'}/\text{ms2}\}$
18	$\{p, e, r, s\}$	$\{\text{rp1}\}$
19	$\{p, h\}$	$\{\text{oaft1}, \text{ms1}, \text{oaft2}, {}^{u_2}/\text{ms2}\}$
20	$\{r\}$	$\{\text{oaft1}, \text{co1}, \text{rp1}, {}^{u_3}/\text{oaft2}, \text{co2}, \text{rp2}\}$
21	$\emptyset$	$\{\text{oaft1}, \text{co1}, \text{rp1}, \text{ms1}, \text{ba1}, \text{oaft2}, \text{co2}, \text{rp2}, \text{ms2}, \text{ba2}\}$
22	$\{p, m, h, {}^{u_4}/i, {}^{u_3}/r\}$	$\{\text{oaft2}\}$
23	$\{p, {}^{u_3}/r\}$	$\{\text{oaft1}, \text{rp1}, {}^{u_3'}/\text{ms1}, \text{oaft2}, {}^{u_3'}/\text{ms2}\}$
24	$\{{}^{u_2}/h\}$	$\{\text{oaft1}, \text{co1}, {}^{u_2'}/\text{rp1}, \text{ms1}, {}^{u_2'}/\text{ba1}, \text{oaft2}, {}^{u_2'}/\text{co2}, {}^{u_2'}/\text{rp2}, \text{ms2}, {}^{u_2'}/\text{ba2}\}$

Consider a crisply generated $\mathbf{L}$-concept $\langle\{p, h, {}^{u_4}/i, {}^{u_3}/r\}, \{\text{oaft1}, {}^{u_4'}/\text{ms1}, \text{oaft2}, {}^{u_2 \wedge u_4'}/\text{ms2}\}\rangle$ and an admissible assignment v with $v(u_1) = v(u_3) = v(u_4) = 0$ and $v(u_2) = 1$.
Then
$\bar{v}^{\mathcal{B}_c(X,Y,I)}(\langle\{p, h, {}^{u_4}/i, {}^{u_3}/r\}, \{\text{oaft1}, {}^{u_4'}/\text{ms1}, \text{oaft2}, {}^{u_2 \wedge u_4'}/\text{ms2}\}\rangle)$
$= \langle \bar{v} \circ \{p, h, {}^{u_4}/i, {}^{u_3}/r\}, \bar{v} \circ \{\text{oaft1}, {}^{u_4'}/\text{ms1}, \text{oaft2}, {}^{u_2 \wedge u_4'}/\text{ms2}\}\rangle$
$= \langle\{p, h\}, \{\text{oaft1}, \text{oaft2}, \text{ms1}, \text{ms2}\}\rangle$ is a formal concept of a completion $\langle X, Y, \bar{v} \circ I\rangle$.

References

1. Krupka M., Lastovicka J.: Concept lattices of incomplete data. In: F. Domenach, D.I. Ignatov, and J. Poelmans (Eds.): ICFCA 2012, LNAI 7278. Springer, Heidelberg, 2012, pp. 180194.
2. Krupka M., Lastovicka J.: Fuzzy concept lattices with incomplete knowledge. In: S. Greco, B. Bouchon-Meunier, G. Coletti, M. Fedrizzi, B. Matarazzo, R. R. Yager (Eds.): Advances in Computational Intelligence, 14th International Conference on Information Processing and Management of Uncertainty, IPMU 2012, Catania, Italy, July 9 - July 13, 2012.
3. Ganter, B., Wille, R.: Formal Concept Analysis – Mathematical Foundations. Springer (1999)
4. Belohlavek, R.: Fuzzy Relational Systems: Foundations and Principles. Kluwer Academic Publishers, Norwell, USA (2002)

International Center for Information and Uncertainty
SSIU 2012: International Summer School "Information and Uncertainty"
http://mcin.upol.cz/SSWIU-2012/

european social fund in the czech republic
EUROPEAN UNION
MINISTRY OF EDUCATION, YOUTH AND SPORTS
OP Education for Competitiveness