R. Belohlavek and P. Osicka: Triadic Fuzzy Galois Connections as Ordinary Connections

Department of Computer Science, Palacky University, Olomouc (17. listopadu 12, CZ–77146 Olomouc, Czech Republic)

Motivation

- Triadic Galois connection are fundamental structures behind analysis of relational data
- Representation by ordinary structures allows to automaticaly transfer some of the known results from ordinary case to fuzzy case
- Demonstrate that the idea of using ordinary notions to represent their fuzzy counterparts is useful

Triadic Concept analysis

- Method of analysis of tree-way relational data that aims at extraction of a hierarchically ordered set of clusters
- Growing interest in analysing three-way data. Kolda T. G., Bader B. W.: Tensor decompositions and applications. SIAM Review **51**(3)(2009), 455-500.
- Input: **triadic L-context** = data table $\langle X, Y, Z, I\rangle$ where X, Y, and Z are non-empty sets, and I is a ternary fuzzy relation between X, Y, and Z, i.e. $I : X \times Y \times Z \to L$.
- X, Y, and Z are interpreted as the sets of objects, attributes, and conditions, respectively; $I(x, y, z)$ is interpreted as the degree to which object x has attribute y under condition z. We may also say that $I(x, y, z)$ is the degree to which x, y, z are related.
- For every $\{i, j, k\} = \{1, 2, 3\}$ and a fuzzy set $A_k \in L^{X_k}$, a triadic $\mathbf{L}$-context $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ induces a dyadic $\mathbf{L}$-context

  $$\mathbf{K}^{ij}_{A_k} = \langle X_i, X_j, I^{ij}_{A_k}\rangle$$

  in which the fuzzy relation $I^{ij}_{A_k}$ between X_i and X_j is defined by

  $$I^{ij}_{A_k}(x_i, x_j) = \bigwedge_{x_k \in X_k} (A_k(x_k) \to I\{x_i, x_j, x_k\}) \qquad (1)$$

  for every $x_i \in X_i$ and $x_j \in X_j$.
- The **concept-forming operators** induced by $\mathbf{K}^{ij}_{A_k}$ are ${}^{(i,j,A_k)} : L^{X_i} \to L^{X_j}$ defined by

  $$A_i^{(i,j,A_k)}(x_j) = \bigwedge_{x_i \in X_i} A_i(x_i) \to I^{ij}_{A_k}(x_i, x_j). \qquad (2)$$

- A **triadic L-concept**: a triplet $\langle A_1, A_2, A_3\rangle$ consisting of fuzzy sets $A_1 \in L^{X_1}$, $A_2 \in L^{X_2}$, and $A_3 \in L^{X_3}$, such that for every $\{i, j, k\} = \{1, 2, 3\}$ we have

  $$A_i = A_j^{(i,j,A_k)},\ A_j = A_k^{(j,k,A_i)},\ \text{and}\ A_k = A_i^{(k,i,A_j)}.$$

- The set of all triadic fuzzy concepts contained in the input is called a **concept trilattice**, and it is considered as an output of TCA.

Axiomatization

- We provide axiomatization of concept-forming operators induced by triadic fuzzy contexts
- Triadic $\mathbf{L}$-context $\langle X_1, X_2, X_3, I\rangle$ induces three operators

  $$(i)_I : L^{X_j} \times L^{X_k} \to L^{X_i}$$

  for $\{i, j, k\} = \{1, 2, 3\}$ which are defined by

  $$(A_j, A_k)^{(i)_I} = A_j^{(j,i,A_k)} \qquad (3)$$

  for any $A_j \in L^{X_j}$ and $A_k \in L^{X_k}$.

Fuzzy Triadic Galois Connection

- Let K be an order filter in $\langle L, \leq\rangle$.
- A **triadic $\mathbf{L}_K$-Galois connection** between sets X_1, X_2, and X_3 is a triplet $\langle {}^{(1)}, {}^{(2)}, {}^{(3)}\rangle$ of mappings

  $${}^{(1)} : L^{X_2} \times L^{X_3} \to L^{X_1},$$
  $${}^{(2)} : L^{X_1} \times L^{X_3} \to L^{X_2},$$
  $${}^{(3)} : L^{X_1} \times L^{X_2} \to L^{X_3},$$

  such that if at least one of the following holds

  $$S(A_3, (A_1, A_2)^{(3)}) \in K,$$
  $$S(A_1, (A_2, A_3)^{(1)}) \in K,$$
  $$S(A_2, (A_1, A_3)^{(2)}) \in K,$$

  then

  $$S(A_3, (A_1, A_2)^{(3)}) = S(A_1, (A_2, A_3)^{(1)}) = S(A_2, (A_1, A_3)^{(2)}).$$

- Alternative definition: for $\langle {}^{(1)}, {}^{(2)}, {}^{(3)}\rangle$ the following holds
  - (a) $A_i \subseteq (A_j, (A_i, A_j)^{(k)})^{(i)}$ (extensivity),
  - (b) if $S(A_i, A'_i) \in K$ then $S(A_i, A'_i) \leq S((A'_i, A_j)^{(k)}, (A_i, A_j)^{(k)})$ (antitony).
- **One-to-one correspondence to ternary fuzzy relations**: Let $I \in L^{X_1 \times X_2 \times X_3}$. Let $\langle {}^{(1)}, {}^{(2)}, {}^{(3)}\rangle$ be a triadic $\mathbf{L}_L$-Galois connection between X_1, X_2, and X_3 and define a ternary relation $I_{\langle {}^{(1)}, {}^{(2)}, {}^{(3)}\rangle} \in L^{X_1 \times X_2 \times X_3}$ by

  $$I_{\langle {}^{(1)}, {}^{(2)}, {}^{(3)}\rangle}(x_1, x_2, x_3) = (\{1/x_1\}, \{1/x_2\})^{(3)},$$

  then
  - (a) the triplet $\langle {}^{(1)_I}, {}^{(2)_I}, {}^{(3)_I}\rangle$ forms a triadic $\mathbf{L}_L$-Galois connection,
  - (b) $I = I_{\langle {}^{(1)_I}, {}^{(2)_I}, {}^{(3)_I}\rangle}$,
  - (c) $\langle {}^{(1)}, {}^{(2)}, {}^{(3)}\rangle = \langle {}^{(1)_{I_{\langle (1),(2),(3)\rangle}}}, {}^{(2)_{I_{\langle (1),(2),(3)\rangle}}}, {}^{(3)_{I_{\langle (1),(2),(3)\rangle}}}\rangle$.

Cartesian representation

- We utilize the following mappings. For a fuzzy set $A \in L^U$ put

  $$\lfloor A\rfloor = \{\langle u, a\rangle \in U \times L \mid a \leq A(u)\};$$

  For an ordinary set $B \subseteq U \times L$, define a fuzzy set $\lceil B\rceil$ in U by

  $$\lceil B\rceil(u) = \bigvee_{\langle u,a\rangle \in B} a.$$

  $\lfloor A\rfloor$ may be thought of as the area below A while $\lceil B\rceil$ may be thought of as an upper envelope of B.
- An (ordinary) triadic Galois connection $\langle {}^{\langle 1\rangle}, {}^{\langle 2\rangle}, {}^{\langle 3\rangle}\rangle$ between $X_1 \times L$, $X_2 \times L$, $X_3 \times L$ is called **commutative with respect to $\lfloor\lceil\rceil\rfloor$** iff

  $$(\lfloor\lceil A_i\rceil\rfloor, \lfloor\lceil A_j\rceil\rfloor)^{\langle k\rangle} = \lfloor\lceil (A_i, A_j)\rceil\rfloor^{\langle k\rangle} \qquad (4)$$

  holds for any $\{i, j, k\} = \{1, 2, 3\}$ and any sets $A_1 \in X_1 \times L$, $A_2 \in X_2 \times L$, and $A_3 \in X_3 \times L$.
- For a triadic Galois connection $\langle {}^{\langle 1\rangle}, {}^{\langle 2\rangle}, {}^{\langle 3\rangle}\rangle$ between $X_1 \times L$, $X_2 \times L$, $X_3 \times L$, and fuzzy sets $A_i \in L^{X_i}$, $A_j \in L^{X_j}$, and $A_k \in L^{X_k}$ we define mappings ${}^{(i)_{\langle i\rangle}} : L^{X_j} \times L^{X_k} \to L^{X_i}$ by

  $$(A_j, A_k)^{(i)_{\langle i\rangle}} = \lceil(\lfloor A_j\rfloor, \lfloor A_k\rfloor)^{(i)}\rceil \qquad (5)$$

- Let $\langle {}^{(1)}, {}^{(2)}, {}^{(3)}\rangle$ be a triadic $\mathbf{L}$-Galois connection between X_1, X_2, and X_3. Then for sets $A_i \in X_i \times L$, $A_j \in X_j \times L$, and $A_k \in X_k \times L$, we define mappings ${}^{\langle i\rangle_{(i)}} : (X_j \times L) \times (X_k \times L) \to X_i \times L$ by

  $$(A_j, A_k)^{\langle i\rangle_{(i)}} = \lfloor(\lceil A_j\rceil, \lceil A_k\rceil)^{(i)}\rfloor \qquad (6)$$

- **Representation theorem**
  Let $\langle {}^{(1)}, {}^{(2)}, {}^{(3)}\rangle$ be a triadic $L_{\{1\}}$-Galois connection between X_1, X_2, and X_3 and $\langle {}^{\langle 1\rangle}, {}^{\langle 2\rangle}, {}^{\langle 3\rangle}\rangle$ be a triadic Galois connection between $X_1 \times L$, $X_2 \times L$, and $X_3 \times L$. Then the following holds:
  - (a) $\langle {}^{\langle 1\rangle_{(1)}}, {}^{\langle 2\rangle_{(2)}}, {}^{\langle 3\rangle_{(3)}}\rangle$ is a triadic Galois connection commutative with respect to $\lfloor\lceil\rceil\rfloor$.
  - (b) $\langle {}^{(1)_{\langle 1\rangle}}, {}^{(2)_{\langle 2\rangle}}, {}^{(3)_{\langle 3\rangle}}\rangle$ is a triadic $\mathbf{L}_{\{1\}}$-Galois connection.
  - (c) The map $\langle {}^{(1)}, {}^{(2)}, {}^{(3)}\rangle \mapsto \langle {}^{\langle 1\rangle}, {}^{\langle 2\rangle}, {}^{\langle 3\rangle}\rangle$ is an one-to-one map between the set of all triadic $\mathbf{L}_{\{1\}}$-Galois connections between X_1, X_2, and X_3 and the set of all triadic Galois connections between $X_1 \times L$, $X_2 \times L$, $X_3 \times L$ that are commutative with respect to $\lfloor\lceil\rceil\rfloor$.
- Application: alowing results known from the ordinary setting to be transfered the fuzzy setting, e.g. theorem saying that every fuzzy concept trilattice is isomorphic to some ordinary concept trilattice via a natural isomorphism

Cut-like representation

- Inspired by the notion of an a-cut of a fuzzy set
- An a-**cut** of fuzzy set $A \in L^U$ is the ordinary subset of U defined by

  $${}^aA = \{u \in U \mid a \leq A(u)\}.$$

- A system $\{\langle {}^{(1_a)}, {}^{(2_a)}, {}^{(3_a)}\rangle \mid a \in L\}$ of (ordinary) triadic Galois connections is called **L-nested** iff
  - (a) for each $a, b \in L$ such that $a \leq b$, and $A_i \in L^{X_i}$, $A_j \in L^{X_j}$ it holds $(A_i, A_j)^{(k_a)} \supseteq (A_i, A_j)^{(k_b)}$
  - (b) for all $x_i \in X_i$, $x_j \in X_j$, $x_k \in X_k$ the set $\{a \in L \mid x_i \in (\{x_j\}, \{x_k\})^{(i_a)}\}$ has a greatest element.
- For a triadic $\mathbf{L}$-Galois connection $\langle {}^{(1)}, {}^{(2)}, {}^{(3)}\rangle$ between X_1, X_2, and X_3 denote

  $$\mathcal{C}_{\langle (1),(2),(3)\rangle} = \{\langle {}^{(1_a)}, {}^{(2_a)}, {}^{(3_a)}\rangle \mid a \in L\}.$$

- For an $\mathbf{L}$-nested system $\{\langle {}^{(1_a)}, {}^{(2_a)}, {}^{(3_a)}\rangle \mid a \in L\}$ of triadic Galois connections between X_1, X_2, and X_3 denote by $\langle {}^{(1)_\mathcal{C}}, {}^{(2)_\mathcal{C}}, {}^{(3)_\mathcal{C}}\rangle$ the mappings defined for $\{i, j, k\} = \{1, 2, 3\}$, and $A_i \in L^{X_i}$, $A_j \in L^{X_j}$ by

  $$(A_i, A_j)^{(k)_\mathcal{C}}(x_k) = \bigvee\{a \mid x_k \in \bigcap_{b,c \in L} ({}^bA_i, {}^cA_j)^{(k_{a\otimes b\otimes c})}\}.$$

- **Representation theorem**
  - (a) $\mathcal{C}_{\langle (1),(2),(3)\rangle}$ is an $\mathbf{L}$-nested system of triadic Galois connections,
  - (b) $\langle {}^{(1)_\mathcal{C}}, {}^{(2)_\mathcal{C}}, {}^{(3)_\mathcal{C}}\rangle$ is a triadic $\mathbf{L}$-Galois connection,
  - (c) $\langle {}^{(1)}, {}^{(2)}, {}^{(3)}\rangle = \langle {}^{(1)_{\mathcal{C}_{\langle (1),(2),(3)\rangle}}}, {}^{(2)_{\mathcal{C}_{\langle (1),(2),(3)\rangle}}}, {}^{(3)_{\mathcal{C}_{\langle (1),(2),(3)\rangle}}}\rangle$, and $\mathcal{C} = \mathcal{C}_{\langle (1)_\mathcal{C},(2)_\mathcal{C},(3)_\mathcal{C}\rangle}$, i.e. the mappings between the sets of all triadic $\mathbf{L}$-Galois connections and all nested systems of triadic Galois connections are mutually inverse bijections.

Summary

- Axiomatic characterization of fuzzy triadic galois connections
- Two ways to represent them by ordinary Galois connections

International Center for Information and Uncertainty
SSIU 2012: International Summer School "Information and Uncertainty"
http://mcin.upol.cz/SSWIU-2012/

european social fund in the czech republic
EUROPEAN UNION
MINISTRY OF EDUCATION, YOUTH AND SPORTS
OP Education for Competitiveness