Lucie Urbanova, Vilem Vychodil: Derivation Digraphs for Graded If-Then Rules

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Introduction

We present a graph-based method of reasoning with if-then rules describing dependencies between graded attributes. The rules are usually written as expressions $A \Rightarrow B$, where A and B are fuzzy sets of attributes, and have two basic interpretations:

:: Attribute implications in formal fuzzy context:
"If it is (very) true that an object has all attributes from A, then it has also all attributes from B."

:: Fuzzy functional dependencies in similarity-based relational databases:
"For any two objects (rows): if they have (very) similar values on attributes from A then they have similar values on attributes from B".

What those two interpretations have in common?

:: same syntax (although different interpretation)

:: same notion of semantic entailment

:: single inference system can be used

Armstrong-like axiomatic system ($\mathbf{L}$ finite, Y set of attributes, $A, B, C, D \in \mathbf{L}^Y$):

(Ax) infer $A \cup B \Rightarrow B$
(Cut) from $A \Rightarrow B$ and $B \cup C \Rightarrow D$ infer $A \cup C \Rightarrow D$
(Mul) from $A \Rightarrow B$ infer $c^* \otimes A \Rightarrow c^* \otimes B$,

Two kind of completeness can be shown:

:: Ordinary completeness

- Theory T - set of if-then rules
- $A \Rightarrow B$ is provable from T iff $A \Rightarrow B$ semantically follows from T (in degree 1)

:: Graded (Pavelka-style) completeness

- Theory T - fuzzy set of if-then rules
- A degree to which $A \Rightarrow B$ is provable from T equals to degree to which $A \Rightarrow B$ semantically follows from T.

Maier (2) proposed graph-based approach in order to normalize proof for reasoning with (ordinary) functional dependencies in relational databases. Can this approach be extended for graded if-then rules? YES.

Derivation Digraphs for graded if-then rules

:: acyclic digraphs

:: vertices are attributes from Y

:: arcs correspond to if-then rules used from an input theory

:: construction:
every set of unconnected vertices is derivation diagram
new vertex for attribute y is added to graph if the validity of y resulting from T is strictly higher then the degree y has in the current state of graph, formally:

$\boldsymbol{T}$-based $\mathbf{L}^*$-derivation diagram (DAG):
Let $\mathbf{L}^* = \langle L, \wedge, \vee, \otimes, \rightarrow, {}^*, 0, 1\rangle$ be be complete residuated lattice with a truth-stressing hedge, T be a set of graded if-then rules over Y.

1) Any $\mathbf{D} = \langle V, \emptyset\rangle$ such that $\emptyset \neq V \subseteq Y \times L$ is a T-based $\mathbf{L}^*$-derivation DAG;
2) If $\mathbf{D} = \langle V, A\rangle$ is a T-based $\mathbf{L}^*$-derivation DAG and there are $E \Rightarrow F \in T$, attribute $y \in Y$, and vertices $\langle y_1, a_1\rangle \in V, \ldots, \langle y_k, a_k\rangle \in V$ such that for

$$s_0 = \bigwedge\{E(y) \rightarrow 0 \,|\, y \in Y \text{ and } y \notin \{y_1, \ldots, y_k\}\},$$
$$s_1 = \bigwedge\{E(y_i) \rightarrow a_i \,|\, i = 1, \ldots, k\},$$
$$m = \bigvee\{a \in L \,|\, \langle y, a\rangle \in V\},$$
$$d = \big((s_0 \wedge s_1)^* \otimes F(y)\big) \vee m,$$

we have $d > m > 0$, then $\mathbf{D}' = \langle V', A'\rangle$, where

$$V' = V \cup \{\langle y, d\rangle\},$$
$$A' = A \cup \{\langle\langle y_i, a_i\rangle, \langle y, d\rangle\rangle \,|\, i = 1, \ldots, k\},$$

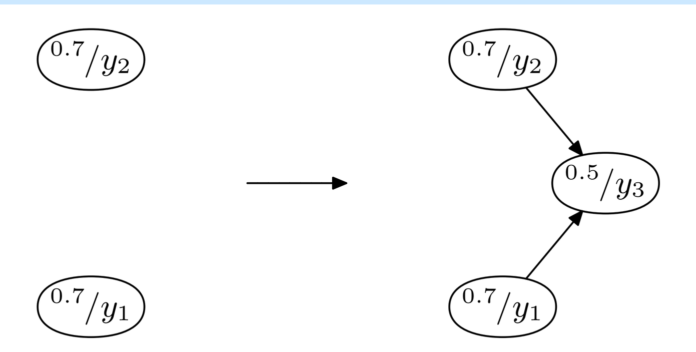
is a T-based $\mathbf{L}^*$-derivation DAG.

One step in construction of DAG

:: $\mathbf{L}^*$: finite linearly ordered Łukasiewicz algebra with $L = \{0, 0.1, \ldots, 0.9, 1\} \subseteq [0, 1]$, $\wedge$ and $\vee$ usual minima and maxima, $\otimes, \rightarrow$ Łukasiewicz operations, * identity.

:: set of initial vertices: $\{\langle y_1, 0.7\rangle, \langle y_2, 0.7\rangle\}$
theory: $T = \{\ldots, \{{}^{0.7}\!/y_1, {}^{0.8}\!/y_2\} \Rightarrow \{{}^{0.6}\!/y_3\}, \ldots\}$

:: adding vertex $\langle 0.5, y_3\rangle$



$s_0 = 1$
$s_1 = 0.9$
$m = 0$
$d = (0.9 \otimes 0.6)^*$
$= 0.5$

Completeness

For T-based $\mathbf{L}^*$-derivation DAG $\mathbf{D}$ we have:

:: yield of $\mathbf{D}$ on attribute y: $\mathbf{D}(y) = \bigvee\{a \in L \,|\, \langle y, a\rangle \in V\}$

:: $\mathbf{D}$ is called a T-based $\mathbf{L}^*$-derivation DAG for $E \Rightarrow F$ if $\{\langle y, E(y)\rangle \,|\, y \in Y \text{ and } E(y) > 0\}$ is the set of initial vertices of $\mathbf{D}$ and $\mathbf{D}(y) \geq F(y)$ for all $y \in Y$.

Results

Let T be a theory (set of if-then rules)

:: If $T \vdash A \Rightarrow B$ (using (Ax),(Cut),(Mul)), then there is a T-based $\mathbf{L}^*$-derivation DAG for $A \Rightarrow B$.

:: If there is a T-based $\mathbf{L}^*$-derivation DAG for $A \Rightarrow B$, then $T \vdash A \Rightarrow B$.

:: If $\mathbf{L}$ is finite, then $A \Rightarrow B$ semantically follows from theory T in degree 1 iff there is a T-based $\mathbf{L}^*$-derivation DAG for $A \Rightarrow B$.

:: If $\mathbf{L}$ is finite, then $||A \Rightarrow B||_T$ is the greatest degree $a \in L$ such that there is a T-based $\mathbf{L}^*$-derivation DAG for $A \Rightarrow a \otimes B$.

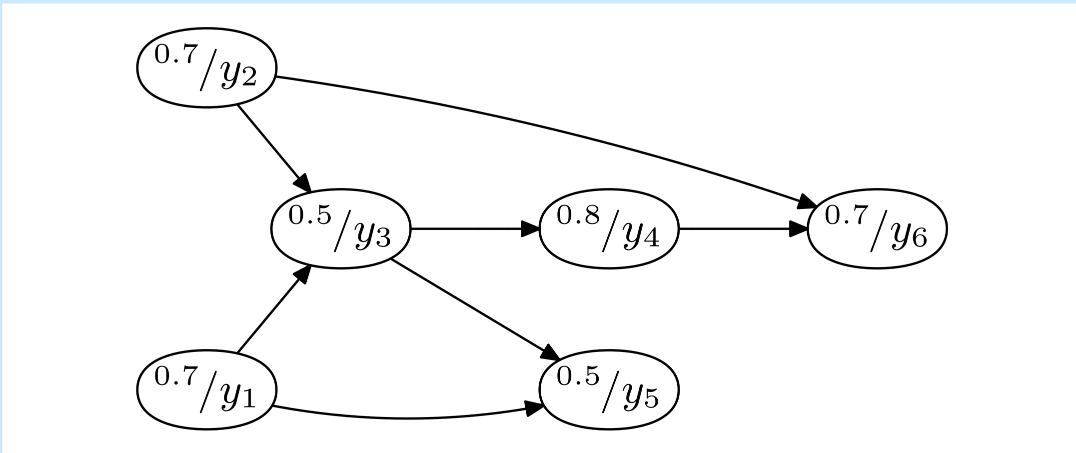
Example

We demonstrate here how to obtain a proof from a T-based $\mathbf{L}^*$-derivation DAG. Assume we have a theory T:

$$T = \{\{{}^{0.7}\!/y_3\} \Rightarrow \{{}^{1}\!/y_4\}, \{{}^{0.7}\!/y_1, {}^{0.8}\!/y_2\} \Rightarrow \{{}^{0.6}\!/y_3\},$$
$$\{{}^{0.6}\!/y_1, {}^{0.9}\!/y_3\} \Rightarrow \{{}^{0.9}\!/y_5, {}^{0.1}\!/y_6\},$$
$$\{{}^{0.5}\!/y_2, {}^{0.6}\!/y_4\} \Rightarrow \{{}^{0.7}\!/y_6\}\}$$

and consider $A = \{{}^{0.7}\!/y_1, {}^{0.7}\!/y_2\}$, $B = \{{}^{0.6}\!/y_4, {}^{0.6}\!/y_6\}$.

Does $||A \Rightarrow B||_T = 1$? YES, since there is T-based $\mathbf{L}^*$-derivation DAG for $A \Rightarrow B$:



:: set of initial vertices is equal to A

:: $B(y) \leq \mathbf{D}(y)$ for all $y \in B$

The following sequence is a part of the corresponding proof for $A \Rightarrow B$ using derivation rules (Ax), (Cut), (Mul) and derived rules projectivity and additivity.

Corresponding Proof

1) $\{{}^{0.7}\!/y_1, {}^{0.7}\!/y_2\} \Rightarrow \{{}^{0.7}\!/y_1, {}^{0.7}\!/y_2\}$ (Ax)
2) $\{{}^{0.7}\!/y_1, {}^{0.8}\!/y_2\} \Rightarrow \{{}^{0.6}\!/y_3\}$ $\in T$
3) $\{{}^{0.6}\!/y_1, {}^{0.7}\!/y_2\} \Rightarrow \{{}^{0.5}\!/y_3\}$ (Mul) on 2)
4) $\{{}^{0.7}\!/y_1, {}^{0.7}\!/y_2\} \Rightarrow \{{}^{0.6}\!/y_1, {}^{0.7}\!/y_2\}$ (Ax)
5) $\{{}^{0.7}\!/y_1, {}^{0.7}\!/y_2\} \Rightarrow \{{}^{0.5}\!/y_3\}$ (Cut)
6) $\{{}^{0.7}\!/y_1, {}^{0.7}\!/y_2\} \Rightarrow \{{}^{0.7}\!/y_1, {}^{0.7}\!/y_2, {}^{0.5}\!/y_3\}$ (Add)
7) $\{{}^{0.7}\!/y_3\} \Rightarrow \{{}^{1}\!/y_4\}$ $\in T$
8) $\{{}^{0.5}\!/y_3\} \Rightarrow \{{}^{0.8}\!/y_4\}$ (Mul) on 7)
......
20) $\{{}^{0.7}\!/y_1, {}^{0.7}\!/y_2\} \Rightarrow \{{}^{0.6}\!/y_4, {}^{0.6}\!/y_6\}$

Conclusions

:: graph-based inference for graded if-then rules

:: degrees of semantic entailment of fuzzy attribute implication (fuzzy functional dependency) from a theory - fuzzy set of fuzzy attribute implication (fuzzy functional dependencies) can be characterized by the existence of directed acyclic graphs

Future work

:: further properties of diagram, complexity issues

:: characterization of syntactic closures via DAG

:: algorithm for computing closures

References

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