Quantum Structures I-III

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- observable: $f : \Omega \to \mathbb{R}$, s.t. $f^{-1}(E) \in \mathcal{S}$, $E \in \mathcal{B}(\mathbb{R})$ - measurable

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- $x_t := x((-\infty, t))$, $t \in \mathbb{R}$
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- $$f(\omega) = \begin{cases} \inf\{r_j : \omega \in x_{r_j}\} & \text{if } \omega \in \bigcup_n A_n, \\ 0 & \text{if } \omega \notin \bigcup_n A_n. \end{cases}$$

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- for classical mechanics

$$\inf_s(\sigma_s(x)\sigma_s(y)) = 0.$$

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- G. Birkhoff and J. von Neumann, 1936 quantum logic

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Boolean Algebras

A system $A = (A; \vee, \wedge, ', 0, 1)$ is a **Boolean algebra** if type $(2, 2, 1, 0, 0)$ if for all $a, b, c \in A$ we have

1. $a \vee b = b \vee a$, $a \wedge b = b \wedge a$ (commutativity)
2. $(a \vee b) \vee c = a \vee (b \vee c)$, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (associativity)
3. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
   $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (distributivity)
4. $a \vee a' = 1$, $a \wedge a' = 0$
5. $a \wedge 1 = a = a \vee 0$

partial ordering

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- $\leq$ partial ordering on A, (i) $0 \leq a \leq 1$. (ii) $c = a \wedge b$ iff $c \leq a, b$, and if $d \leq a, b$, then $d \leq c$ (greatest lower bound)

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- $(a \vee b)' = a' \wedge b'$, $(a \wedge b)' = a' \vee b'$ (De Morgan)

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- a topological space Ω is **totally disconnected** if
  there exists a base consisting of clopen sets.

Theorem 0.1 (Stone Theorem) *Every Boolean algebra* $A = (A; \vee, \wedge,', 0, 1)$ *is isomorphic to the Boolean algebra of clopen subsets of a compact, totally disconnected Hausdorff topological space (= Stone space).*

Boolean σ-algebras

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  Theorem 0.4 (Loomis-Sikorski) *Every Boolean σ-algebra is a σ-homomorphic image of a σ-algebra of sets.*

  Sketch of the proof:

- Let A be a Boolean σ-algebra and let $\mathcal{A}$ be the algebra of the clopen sets of $\Omega = \text{MaxI}(I)$. For $a \in A$, let $h(A) = a$. If $\{a_n\}$ and $\{A_n\}$, then if $a = \bigvee_n a_n$ and $h(a) = A$, we have $A \supseteq \bigcup_n A_n$, and $A \setminus \bigcup_n A_n$ is a meager set.

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- $\hat{h}$ is a σ-homomorphism of $\mathcal{S}$ onto A.

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- Connection with basically disconnected spaces:
- X is said to be *basically disconnected* provided the closure of every open F_σ subset of X is open.

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(v) $b = a \vee (b \wedge a^{\perp})$ whenever $a \leq b$ (orthomodular law).

- H- Hilbert space,

  $L(H) = \{M \subseteq H : M-$closed subspace of $H\}$

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- Gleason's Theorem, $2 < \dim H \leq \aleph_0$

- S-prehilbert space

$$E(S) = \{M \subseteq S : \; M + M^{\perp} = S\}$$

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- A, B hermitian operators are compatible iff $AB = BA$

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- finite sequence $\{B_0, \dots, B_{n-1}\}$ from $\mathcal{B}$ is a loop of order n $(n \geq 3)$ if
- (i) $\forall\ i \in \{0, 1, \dots, n-1\}$ we have $B_i \cap B_{i+1} = \{0, 1, x, x^\perp\}$ x atom in both BAs

- (ii) if $j \notin \{i-1, i, i+1\}$, $B_i \cap B_j = \{0, 1\}$

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- $a + b$ exists, then so does $a \vee b$, and $a + b = a \vee b$
- or iff $a + b$, $b + c$ and $a + c$ exist, then $a + b + c$ is defined in A

Firefly Examples of quantum structures

Fig. 4.1

Firefly Examples of quantum structures

Fig. 4.1

- The experiment A: Look at the front window.
  The experiment B: Look at the side window.
  The outcomes of A and B are:

- See a light in the left half (l_A, l_B), right half (r_A, r_B) of the window or see no light (n_A, n_B). It is clear that $n_A = n_B =: n$ and we put $l_A =: l$, $r_A =: r$, $l_B =: f$, $r_B =: b$ (f for the front, b for the back)

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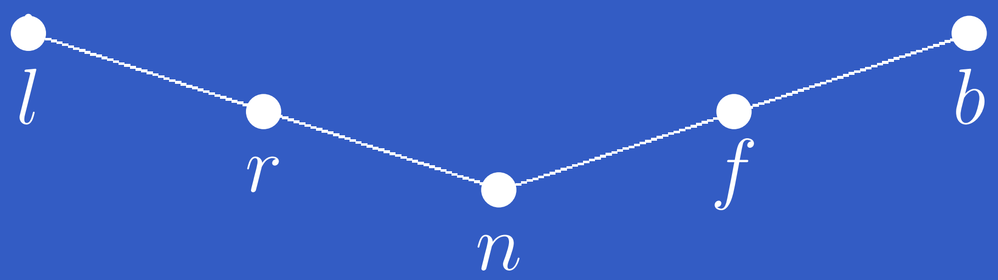
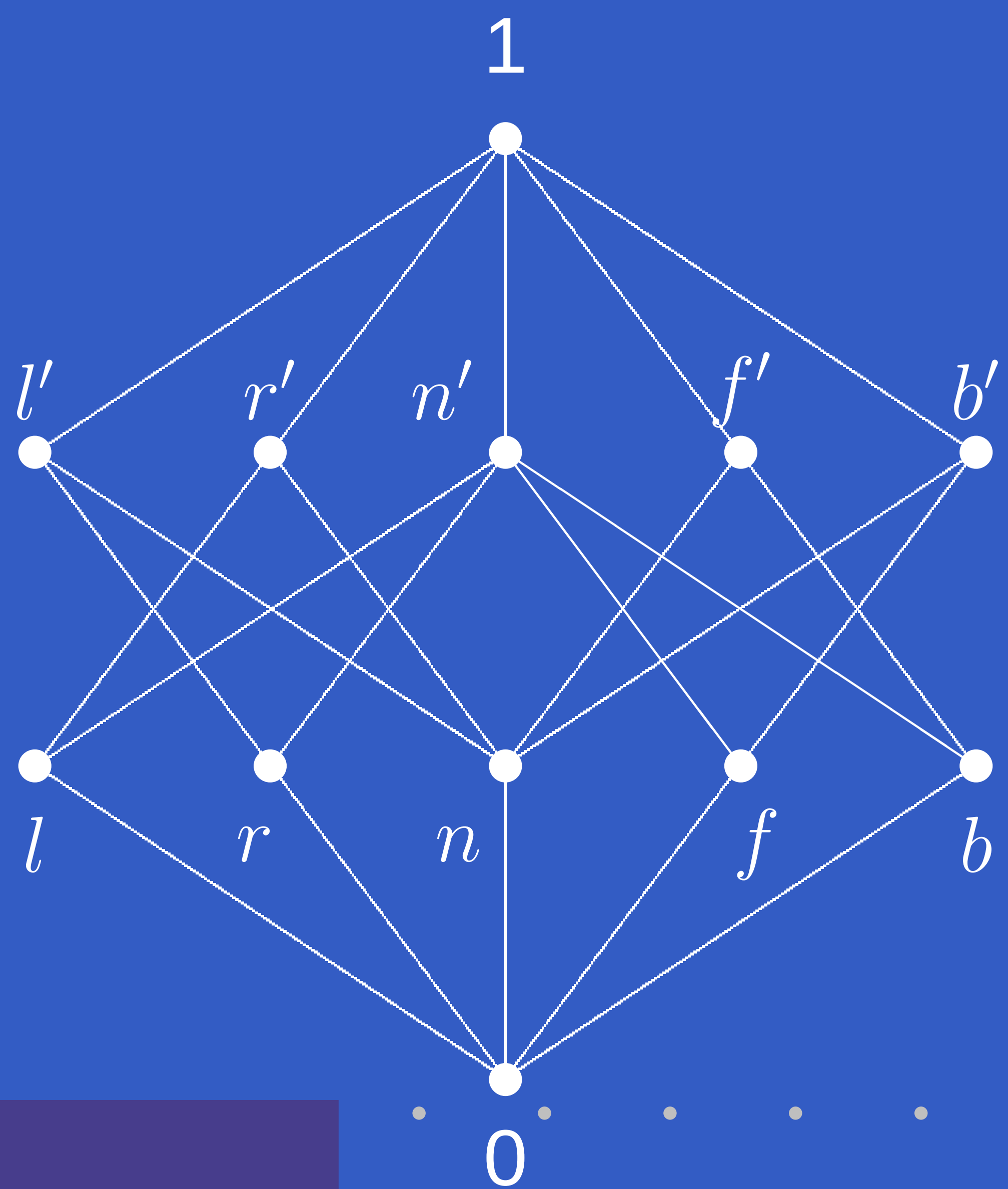


Fig. 4.2


1
l'
r'
n'
f'
b'
l
r
n
f
b
0

Three-chamber box

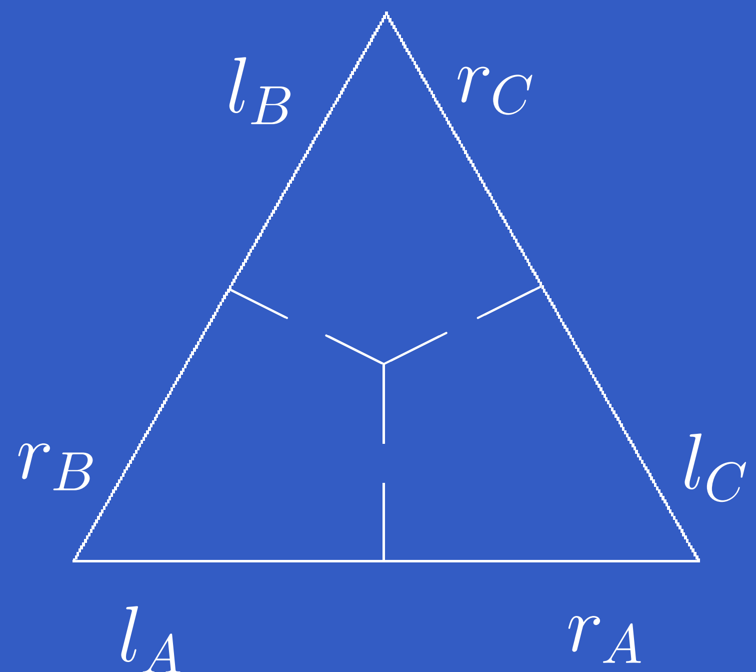


Fig. 4.5

Three-chamber box

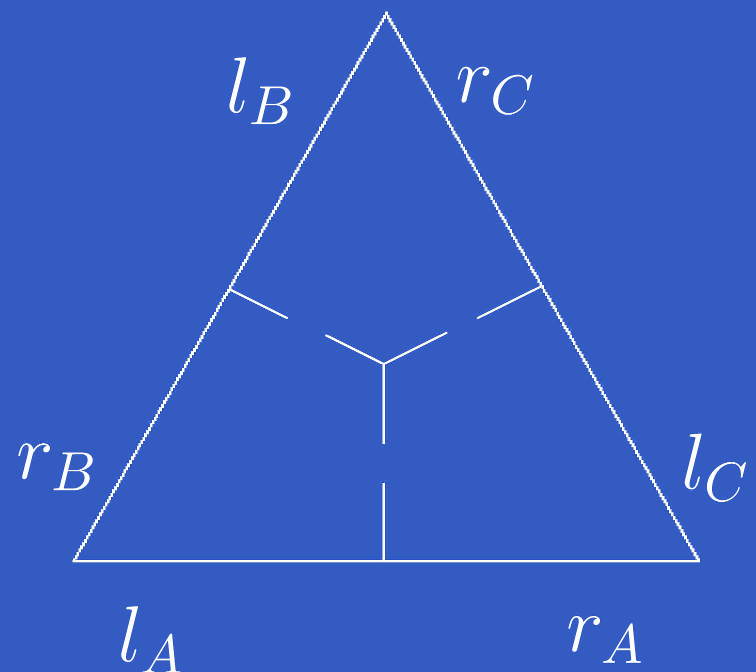


Fig. 4.5

- three experiments, corresponding to the three windows A, B and C. we record l_E, r_E, n_E if we see, respectively, a light to the left, right, of the center line or no light.

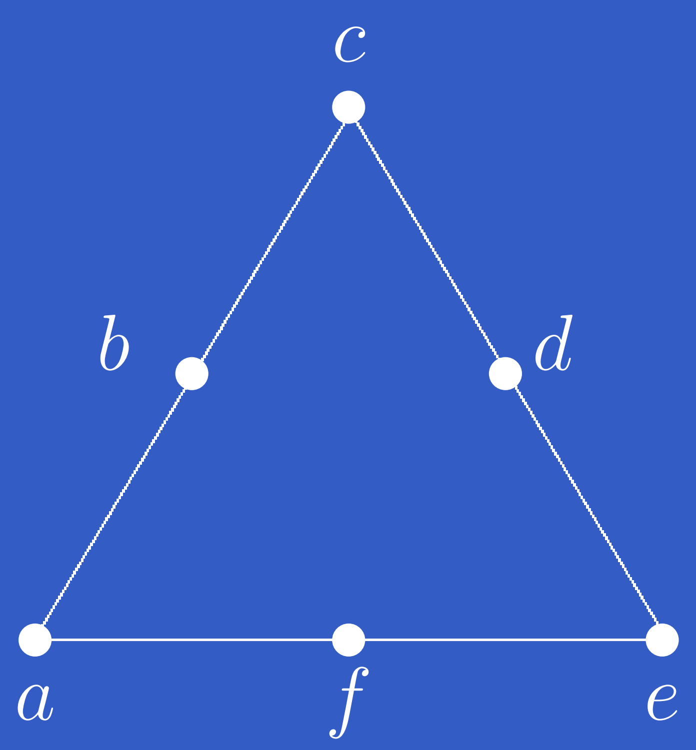
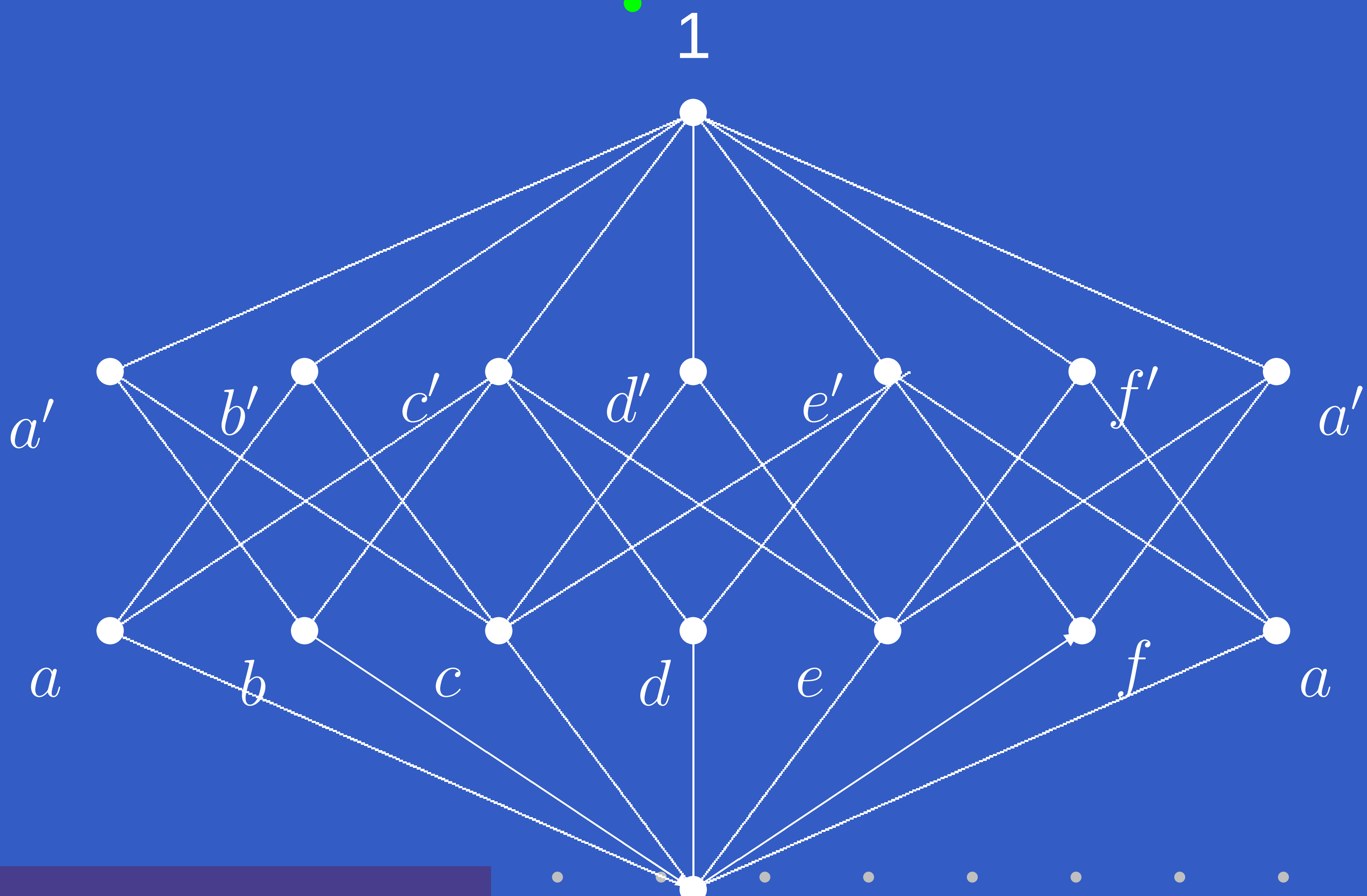


Fig. 4.6 Wright triangle


1
a'
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Examples

$[0,1]$ + restricted from $[0,1]$

po-group $(G; \leq, +, -, 0)$

$$a \leq b \quad \rightarrow \quad a + c \leq b + c$$

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interval EA: $E := \Gamma(G, u)$

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interval EA: $E := \Gamma(G,u)$

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$s(1) = 1.$

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- G - interpolation group whenever $a_1, a_2 \le b_1, b_2 \; \exists \; c \in G$ s.t. $a_1, a_2 \le c \le b_1, b_2$

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- Ulam-game, Pinocchio, Game with black-and white marbles, error correcting codes.
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MV-algebra is an algebra $M = (M; \oplus, \odot, ^*, 0, 1)$ of type (2,2,1,0,0) such that, for all $a, b, c \in M$, we have

(i) $a \oplus b = b \oplus a$;

(ii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;

(iii) $a \oplus 0 = a;$

(iv) $a \oplus 1 = 1;$

(v) $(a^*)^* = a;$

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1. $a \vee b = (a^* \oplus b)^* \oplus b$. M is a distributive lattice

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- **Bold algebra** $\mathcal{F} \subseteq [0,1]^\Omega$ (i) $1 \in \mathcal{F}$, (ii) $f \in \mathcal{F}$, then $1 - f \in \mathcal{F}$, (iii) $f, g \in \mathcal{F}$, and

$$(f \oplus g)(\omega) := \min\{f(\omega) + g(\omega), 1\}, \omega \in \Omega,$$

then $f \oplus g \in \mathcal{F}$.

$$(f \odot g)(\omega) := \max\{0, (f(\omega) + g(\omega) - 1)\}$$

- Let $(G, +, 0, \leq)$ be an ℓ-group, i.e. a group such that if $a \leq b$, $a, b \in G$, then for any $c \in G$, $c + a \leq c + b$, and G is a lattice.

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- $\Gamma(G, u) = [0, u]$

$$a \oplus b = (a + b) \wedge u, a, b \in \Gamma(G, u),$$

$$a \odot b = 0 \vee (a + b - u), a, b \in \Gamma(G, u)$$

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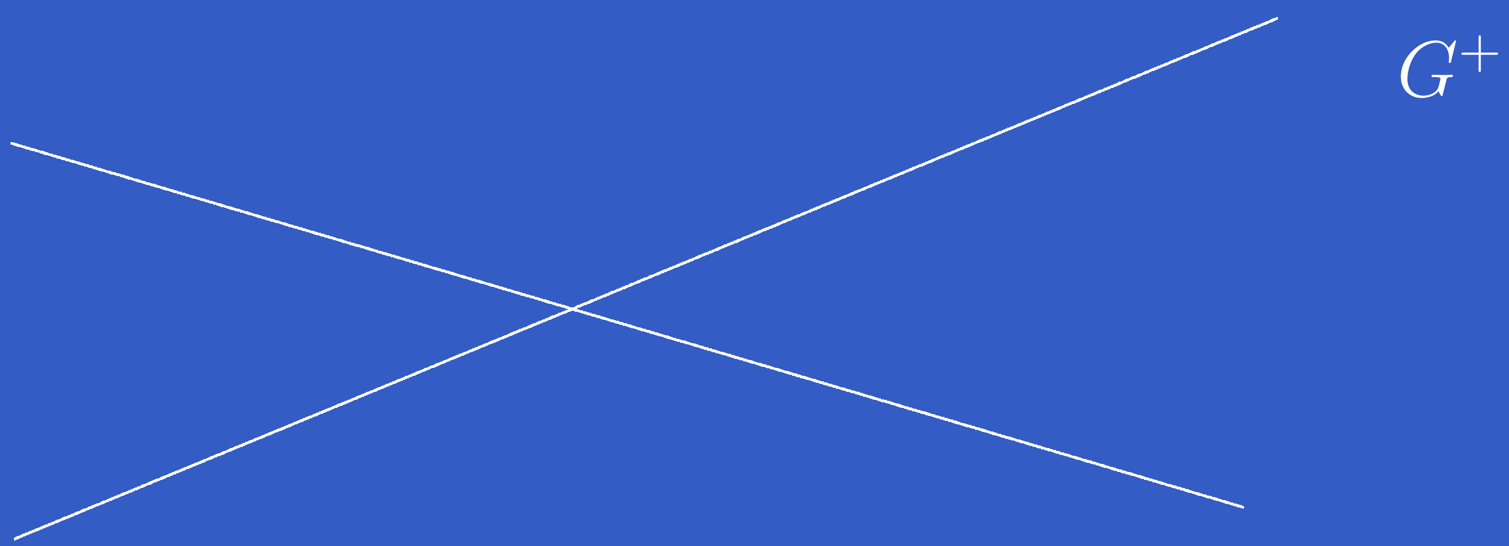

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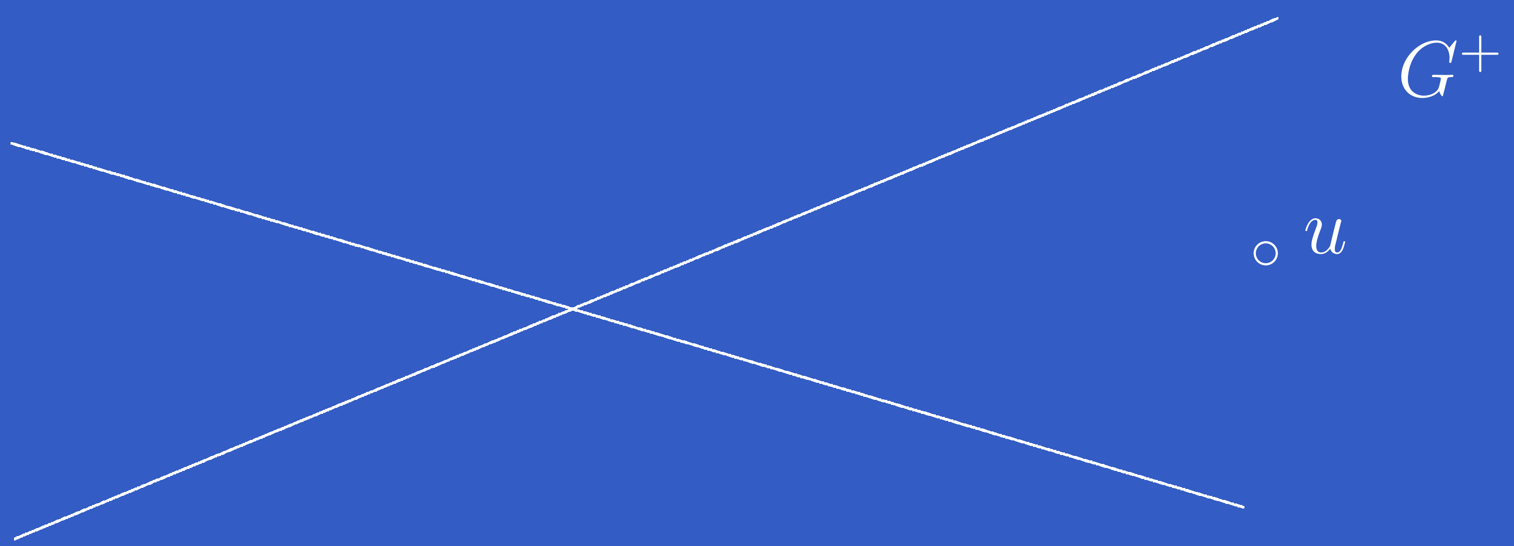

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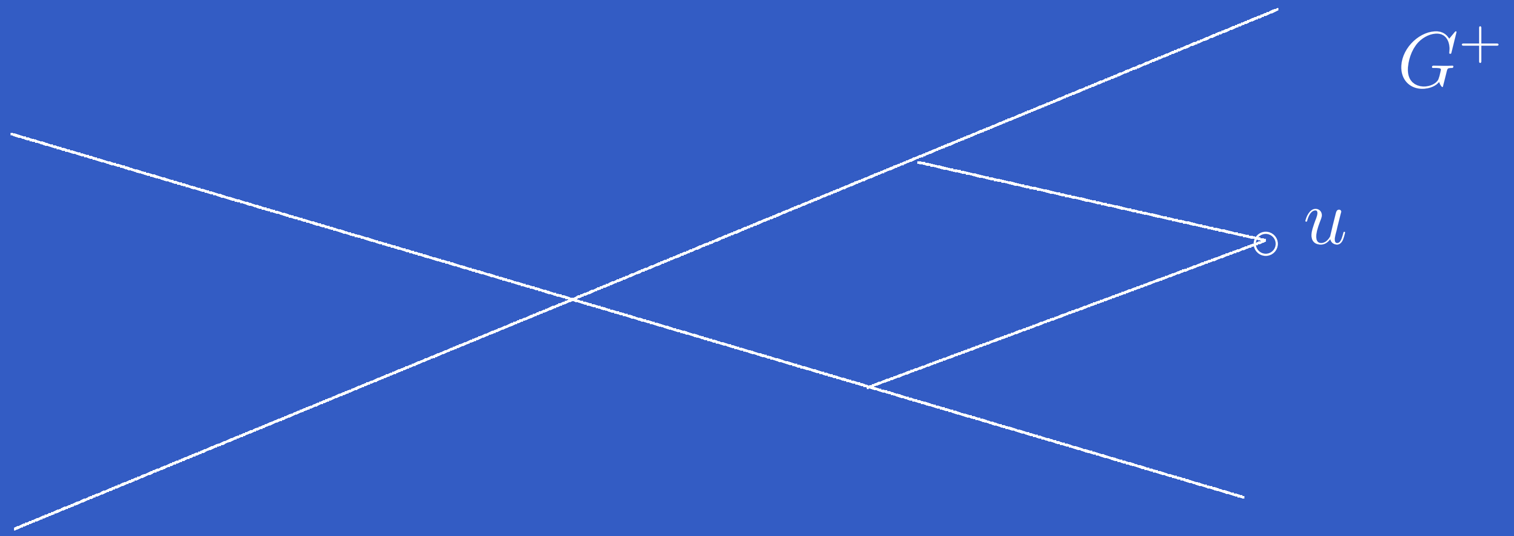

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- $\mathcal{S}(E) \cong \mathcal{S}(Aff(\mathcal{S}(E)), 1)$, $s \mapsto f(s)$, $f \in$ Aff$(\mathcal{S}(E))$

Simplices vs EAs

- convex cone- in a real linear space V is any subset C of V such that (i) $0 \in C$, (ii) if $x_1, x_2 \in C$, then $\alpha_1 x_1 + \alpha_2 x_2 \in C$ for any $\alpha_1, \alpha_2 \in \mathbb{R}^+$.

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- base- for a convex cone C is any convex subset K of C $y \in C \setminus \{0\}$ may be uniquely expressed in the form $y = \alpha x$ for some $\alpha \in \mathbb{R}^+$, $x \in K$

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- $\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & 1-\beta_1 \end{pmatrix}$, the parameters β_1 and β_2 must satisfy the inequality $(\beta_1 - \frac{1}{2})^2 + \beta_2^2 \leq \frac{1}{4}$, and vice-versa. Hence, the state space is affinely isomorphic with the latter circle. The state space for $H = \mathbb{C}^2$ is affinely homeomorphic with a three-dimensional real sphere

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- **Theorem 0.11** *Let E be an effect algebra with RDP and let s be a state on E. Then there is a unique maximal regular Borel probability measure $\mu_s \sim \delta_s$ on $\mathcal{B}(\mathcal{S}(E))$ such that*

$$s(a) = \int_{\mathcal{S}(E)} \hat{a}(x)\, \mathsf{d}\mu_s(x), \quad a \in E.$$

- **Theorem 0.12** *Let $E = \Gamma(G, u)$ be an interval effect algebra where (G, u) is a unigroup, and let $\mathcal{S}(E)$ be a simplex. If s is σ-additive, then its unique extension, $\hat{s}$, on (G, u) is σ-additive.*

- **Theorem 0.13** *Let E be an MV-algebra and let s be a state on E. Then there is a unique regular Borel probability measure, μ_s, on $\mathcal{B}(\mathcal{S}(E))$ such that $\mu_s(\partial_e \mathcal{S}(E)) = 1$ and*

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- **Corollary 0.14** *Let s be a state on an effect algebra E. There is a regular Borel probability measure, μ_s, on the Borel σ-algebra $\mathcal{B}(\mathcal{S}(E))$ such that*

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- **Corollary 0.15** *Let s be a state on an effect algebra E. There is a regular Borel probability measure, μ_s, on the Borel σ-algebra $\mathcal{B}(\mathcal{S}(E))$ such that*

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Monotone σ-complete EAs

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- $\mathcal{E}(H)$ is isomorphic to an effect-tribe: $\mathcal{E}(H)$ no RDP
- $\Omega(H) = \{\phi \in H : ||\phi|| = 1\}$, $A \in \mathcal{E}(H)$, $\mu_A(\phi) := (A\phi, \phi)$, $\phi \in \Omega(H)$. $\mathcal{T}(H) = \{\mu_A : A \in \mathcal{E}(H)\}$

Loomis-Sikorski theorems

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Loomis-Sikorski theorems

- **Theorem 0.19** *Every σ-MV-algebra is a σ-homomorphic image of a tribe of fuzzy sets.*
- **Theorem 0.20** *For every monotone σ-complete effect algebra E with RDP, there are a nonempty set Ω, an effect-tribe $\mathcal{T} \subseteq [0,1]^\Omega$ with* RDP*, and a σ-homomorphism h from $\mathcal{T}$ onto E.*

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GMV-algebras

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- PMV-algebra or GMV-algebra is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z;$

(A2) $x \oplus 0 = 0 \oplus x = x;$

(A3) $x \oplus 1 = 1 \oplus x = 1;$

(A4) $1^{\sim} = 0;\ 1^{-} = 0;$

(A5) $(x^{-} \oplus y^{-})^{\sim} = (x^{\sim} \oplus y^{\sim})^{-};$

(A6) $x \oplus (x^{\sim} \odot y) = y \oplus (y^{\sim} \odot x) = (x \odot y^{-}) \oplus y = (y \odot x^{-}) \oplus x;$

(A7) $x \odot (x^{-} \oplus y) = (x \oplus y^{\sim}) \odot y;$

(A8) $(x^{-})^{\sim} = x.$

- $$x \leq y \quad \text{iff} \quad x^- \oplus y = 1$$

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- M – distributive lattice
- $x \vee y = x \oplus (x^\sim \odot y)$ and $x \wedge y = x \odot (x^- \oplus y)$.
- GMV-algebra M is an MV-algebra iff $x \oplus y = y \oplus x$ for all $x, y \in M$.

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$$\begin{aligned} x \oplus y &:= (x + y) \wedge u, \\ x^{-} &:= u - x, \\ x^{\sim} &:= -x + u, \\ x \odot y &:= (x - u + y) \vee 0, \end{aligned}$$

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$(\Gamma(G,u); \oplus, ^{-}, ^{\sim}, 0, u)$ is a GMV-algebra.

- **Theorem 0.21** [Dvu 2002] *For any GMV-algebra M, there exists a unique $($up to isomorphism$)$ unital ℓ-group G with a strong unit u such that $M \cong \Gamma(G, u)$.*
  The functor Γ defines a categorical equivalence between the category of GMV-algebras and the category of unital ℓ-groups.

- **Theorem 0.22** [Dvu 2002] *For any GMV-algebra M, there exists a unique $($up to isomorphism$)$ unital ℓ-group G with a strong unit u such that $M \cong \Gamma(G,u)$.*
  The functor Γ defines a categorical equivalence between the category of GMV-algebras and the category of unital ℓ-groups.
- $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (1,0))$ - GMV-algebra such that $x^\sim = x^-$ (symmetric) but not necessarily MV-algebra

- Let u be the translation $u(t) = t + 1$, $t \in \mathbb{R}$,

  $\mathrm{BAut}(\mathbb{R}) = \{g \in \mathsf{Aut}(\mathbb{R}) : \ \exists\, n \in \mathbb{N}, \ \ u^{-n} \leq g \leq u^n\}.$

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- If $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$.
- If $a + b$ and $a + c$ exist and are equal, then $b = c$. If $b + a$ and $c + a$ exist and are equal, then $b = c$.

Pseudo Effect Algebras

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- $a+b$ and $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist, and in this case, $(a+b)+c = a+(b+c)$.
- If $a+b$ exists, there are elements $d, e \in E$ such that $a+b = d+a = b+e$.
- If $a+b$ and $a+c$ exist and are equal, then $b=c$. If $b+a$ and $c+a$ exist and are equal, then $b=c$.
- If $a+b$ exists and $a+b=0$, then $a=b=0$.

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- PEA is an EA iff $+$ is commutative
- RDP: $a_1 + a_2 = b_1 + b_2$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22}$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$, and $b_2 = c_{21} + c_{22}$.

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- RDP: $a_1 + a_2 = b_1 + b_2$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22}$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$, and $b_2 = c_{21} + c_{22}$.
- (RDP)$_1$: RDP + $x \leqslant c_{12}$ and $y \leqslant c_{21}$, we have $x + y$, $y + x$ exists in E and $x + y = y + x$,

- RDP_2: RDP + $d_2 \wedge d_3 = 0$ - pseudo MV-algebra

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- (G, u) - unital po-group not necessarily Abelian
- AD+Vetterlein: The category of pseudo effect algebras with RDP$_1$ is categorically equivalent with the category of unital po-group with RDP$_1$

States on PEAs

- **Theorem 0.23** *If E is a pseudo effect algebra with* (RDP)*, then either $\mathcal{S}(E)$ is empty or it is a nonempty Choquet simplex.*
  If, in addition, E satisfies (RDP)$_2$*, then either $\mathcal{S}(E)$ is empty or it is a nonempty Bauer simplex.*

States on PEAs

- **Theorem 0.24** *If E is a pseudo effect algebra with* (RDP), *then either $\mathcal{S}(E)$ is empty or it is a nonempty Choquet simplex.*
  If, in addition, E satisfies (RDP)$_2$, *then either $\mathcal{S}(E)$ is empty or it is a nonempty Bauer simplex.*
- Extremal states for GMV-algebras similar as those for MV-algebras

States on PEAs

- **Theorem 0.25** *If E is a pseudo effect algebra with* (RDP)*, then either $\mathcal{S}(E)$ is empty or it is a nonempty Choquet simplex.*
  If, in addition, E satisfies (RDP)$_2$*, then either $\mathcal{S}(E)$ is empty or it is a nonempty Bauer simplex.*
- Extremal states for GMV-algebras similar as those for MV-algebras
- Representation of states by integral as those for states on EAs